

Systematics of IIB spinorial geometry

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Abstract

We reduce the classification of all supersymmetric backgrounds of IIB supergravity to the evaluation of the Killing spinor equations and their integrability conditions, which contain the field equations, on five types of spinors. This extends the work of [hep-th/0503046] to IIB supergravity. We give the expressions of the Killing spinor equations on all five types of spinors. In this way, the Killing spinor equations become a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. This system can be solved to express the fluxes in terms of the geometry and determine the conditions on the geometry of any supersymmetric background. Similarly, the integrability conditions of the Killing spinor equations are turned into a linear system. This can be used to determine the field equations that are implied by the Killing spinor equations for any supersymmetric background. We show that these linear systems simplify for generic backgrounds with maximal and half-maximal number of H -invariant Killing spinors, $H \subset Spin(9,1)$. In the maximal case, the Killing spinor equations factorize, whereas in the half-maximal case they do not. As an example, we solve the Killing spinor equations of backgrounds with two $SU(4) \ltimes \mathbb{R}^8$ -invariant Killing spinors. We also solve the linear systems associated with the integrability conditions of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$ - and $SU(4) \ltimes \mathbb{R}^8$ -backgrounds and determine the field equations that are not implied by the Killing spinor equations.

1 Introduction

Supersymmetric solutions of IIB supergravity have found widespread applications in string theory and gauge theories. Some of these solutions have been discovered in the context of branes, see e.g. [1, 2, 3] and in the context of AdS/CFT correspondence [4], see e.g. [5, 6, 7, 8, 9]. Most of these results rely on Ansätze appropriately adapted to the requirements of the physical problems. Progress has also been made towards a systematic understanding of the supersymmetric solutions of IIB supergravity. The maximally supersymmetric solutions of IIB supergravity have been classified in [10] and they have been found to be locally isometric to Minkowski space, $AdS_5 \times S^5$ [5] and a maximally supersymmetric plane wave [9]. In addition, these backgrounds are related by Penrose limits [11]. More recently, the Killing spinor equations of IIB have been solved for one Killing spinor [12, 13], and for all supersymmetric backgrounds with two $Spin(7) \ltimes \mathbb{R}^8$ -invariant spinors, and four $SU(4) \ltimes \mathbb{R}^8$ - and G_2 -invariant spinors [13].

In the spinorial geometry approach to supersymmetric backgrounds [14], the Killing spinor equations of M-theory and their integrability conditions for any number of supersymmetries turn into linear systems [15]. The linear system of the Killing spinor equations can be solved to express the fluxes of the theory in terms of the geometry and to find the conditions on the geometry imposed by supersymmetry for any number of Killing spinors. The linear system associated with the integrability conditions determines the field equations that are implied by the Killing spinor equations. The main purpose of this paper is to adapt the above results to the Killing spinor equations of IIB supergravity and their integrability conditions. The construction relies on the linearity of the Killing spinor equations and the observation that the IIB Killing spinor equations of any spinor can be determined from those of five types of spinors. These five types of spinors are

$$1, \quad e_{ij}, \quad e_{1234}, \quad e_{i5}, \quad e_{ijk5}, \quad i, j, k = 1, \dots, 4, \quad (1.1)$$

which we denote collectively by σ_I , where we have expressed the spinors in terms of forms. For IIB supergravity, this has been explained in [12], see also appendix A. We evaluate the Killing spinor equations of IIB supergravity on all five types of spinors and express the result in terms of an oscillator basis in the space of spinors. In this way, we can construct a linear system associated with the Killing spinor equations of backgrounds with *any number* of Killing spinors. This linear system can be used to determine the fluxes in terms of the geometry and the conditions on the geometry imposed by supersymmetry. In IIB supergravity, it is first convenient to solve for the complex fluxes, i.e. the three-form field strength G and the one-form field strength P associated with the two scalars. Then the remaining equations determine some of the components of the five-form flux F and constrain the geometry of spacetime.

The Killing spinor equations of supergravity theories imply some of the field equations. In IIB supergravity, this is related to the computation of the field equations from the commutator of supersymmetry transformations [5], see also [21]. We identify the integrability conditions \mathcal{I} and \mathcal{I}_A that contain the field equations and the Bianchi identities¹. Then, we show that the integrability conditions of a Killing spinor can be expressed

¹The Γ -matrix algebra has been carried out using the computer programme GAMMA [22].

in terms of those on five types of spinors σ_I . We evaluate $\mathcal{I}\sigma_I$ and $\mathcal{I}_A\sigma_I$ in terms of an oscillator basis in the space of spinors and thus derive a linear system. This linear system can be used to determine which field equations and Bianchi identities are implied by the Killing spinor equations for backgrounds with any number of Killing spinors.

The main purpose of this paper is to be used as a manual to solve the Killing spinor equations of IIB supergravity for backgrounds with any number of Killing spinors, and to determine the field equations that are implied from the Killing spinor equations for such backgrounds. Because of this, apart from giving the general formulae of the Killing spinor equations acting on all the spinors σ_I , we also list the explicit results in the appendices. From these results, one can construct the linear system associated with the Killing spinor equations of any supersymmetric background. The same applies for the linear system associated with the field equations.

There are several ways to characterize IIB supersymmetric backgrounds. One way is to count the number of Killing spinors² N and their stability subgroup H in $Spin(9, 1) \times U(1)$. The role of the stability subgroup of the Killing spinors in the classification of supersymmetric backgrounds has been stressed in [16]. Backgrounds for which H contains a Berger holonomy group, i.e. H contains $SU(n)$, G_2 , $Sp(2)$ and $Spin(7)$, are of particular interest. The Killing spinors of most of the known solutions have stability subgroups in $Spin(9, 1) \times U(1)$ which are of Berger type. It has been demonstrated in [13] that for any subgroup H in $Spin(9, 1)$, there is a basis in the space of H -invariant spinors Δ^H which can be written as $(\eta_j, i\eta_j)$, $j = 1, \dots, \frac{1}{2}\dim_{\mathbb{R}}\Delta^H$, where η_j are Majorana-Weyl spinors. Moreover it was shown that the Killing spinor equations factorize for some backgrounds which admit $N = \dim_{\mathbb{R}}\Delta^H$ Killing spinors. Here, we shall show that this is the case for *all* backgrounds with $N = \dim_{\mathbb{R}}\Delta^H$ H -invariant Killing spinors, i.e. the *maximally supersymmetric H -backgrounds* or *maximal H -backgrounds*.

In addition, we shall examine the backgrounds that admit $N = \frac{1}{2}\dim_{\mathbb{R}}\Delta^H$ H -invariant Killing spinors, i.e. they admit half of the maximal possible number of H -invariant Killing spinors. We refer to these backgrounds either as *half-maximally supersymmetric H -backgrounds* or as *half-maximal H -backgrounds*. We show that the Killing spinors of such backgrounds can be written as

$$\epsilon = z\eta \tag{1.2}$$

where z is a $N \times N$ complex matrix and η is a basis of N Majorana-Weyl H -invariant spinors. There are two classes of half-maximally supersymmetric H -backgrounds. One class consists of those backgrounds for which the Killing spinors are linearly independent over the complex numbers, and so over the real numbers. Such backgrounds are associated with a complex $N \times N$ invertible matrix z , $\det z \neq 0$. We refer to these models as *generic half-maximal H -backgrounds*. We shall show that although the Killing spinor equations do not factorize in this case, they simplify. In particular, the gravitino Killing spinor equations can be rewritten so that the only contributions that include terms with more than two gamma matrices are those of the F flux. The dependence on the functions of the Killing spinors is also restricted in the terms that contain up to two gamma

²The number of Killing spinors is counted over the real numbers because the Killing spinor equations of IIB supergravity are \mathbb{R} -linear.

matrices. In addition, the solution to the Killing spinor equations gives rise to a parallel transport equation

$$z^{-1}dz + C = 0, \quad (1.3)$$

where C can be interpreted as the restriction of the supercovariant connection on the bundle of Killing spinors \mathcal{K} . This is similar to the parallel transport equation³ that arises in the maximally supersymmetric H -backgrounds [13] but in this case C may depend on z and so the resulting first order system is *non-linear*. The other class consists of those backgrounds for which the Killing spinors are linearly independent over the real numbers but linearly dependent over the complex numbers, so $\det z = 0$. We refer to these models as *degenerate half-maximal H -backgrounds*. Clearly this subclass can be further characterized by the rank of z . If the rank of z is r , then the space of Killing spinors of such backgrounds is of co-dimension $2(N - r)$ in the space of Killing spinors. In particular if the rank of z is $N - 1$, then one of the Killing spinors will be linearly dependent over the complex numbers on the remaining $N - 1$ Killing spinors but linearly independent over the reals.

As an example of our construction, we consider backgrounds with two $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors. The dimension of $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors in the (complex) chiral representation of $Spin(9, 1)$ is four. So backgrounds with two $SU(4) \ltimes \mathbb{R}^8$ -invariant Killing spinors are half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. We solve the Killing spinor equations for both generic and degenerate backgrounds. For generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds, we find that there are two cases to consider. In both cases, we compute the non-linear parallel transport equation $z^{-1}dz + C = 0$ but we do not give the most general solution. Instead, we analyze two examples. In one of the examples z is diagonal with complex entries and in the other z is a real matrix. In both examples, we determine the geometry of the supergravity backgrounds. In particular, we find that the spacetime admits a null Killing vector field and compute all spacetime form bi-linears. In the degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds, the second Killing spinor is proportional to the first Killing spinor, $\epsilon_2 = w\epsilon_1$, where w is a complex function with *non-vanishing* imaginary part, and $\epsilon_1 = (f - g_2 + ig_1)1 + (f + g_2 + ig_1)e_{1234}$ as in [12], $f, g_2 \neq 0$. The geometry of these backgrounds is similar to those with one $SU(4) \ltimes \mathbb{R}^8$ -invariant spinor investigated in [12].

We also solve the linear systems associated with the integrability conditions of the Killing spinor equations of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$ - and $SU(4) \ltimes \mathbb{R}^8$ -backgrounds of [13]. We find that in both cases, if the Bianchi identities are imposed, the only field equations that are not implied by the Killing spinor equations are the E_{--} components of the Einstein equations. We explicitly give these equations in terms of the connection and fluxes of the backgrounds.

This paper is organized as follows. In section two, we review the construction of the bosonic sector of IIB supergravity together with the Killing spinor equations and their integrability conditions. In section three, we show that the Killing spinor equations and the integrability conditions of any spinor can be determined from those on five types of spinors. We also introduce the maximal and half-maximal supersymmetric H -backgrounds and investigate their Killing spinor equations and integrability conditions.

³For maximally supersymmetric backgrounds, $H = 1$ and C is the supercovariant connection.

In section four, we construct the linear systems of the Killing spinor equations and the integrability conditions for any supersymmetric background. In section five, we present two examples of generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. In section six, we solve the Killing spinor equations of degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. In section seven, we solve the linear systems of the integrability conditions of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$ - and $SU(4) \ltimes \mathbb{R}^8$ -backgrounds, and in section eight we give our conclusions. In appendix A, we summarize the construction of spinors in terms of forms and compute various spinor bi-linears. In appendix B, we explicitly give the Killing spinor equations on all five types of spinors. In appendix C, we explicitly give the integrability conditions of the Killing spinor equations on all five types of spinors. In appendix D, we present the linear system of the Killing spinor equations of generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds and give its solution, and in appendix E, we present the linear system of degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds.

2 Killing spinor equations and integrability conditions

2.1 Killing spinor equations

The bosonic fields of IIB supergravity [17, 5, 18] are the spacetime metric g , two real scalars, the axion σ and the dilaton ϕ , which are combined into a complex one-form field strength P , two three-form field strengths G^1 and G^2 which are combined to a (complex) three-form field strength G , and a self-dual five-form field strength F . To describe these⁴, we introduce a $SU(1, 1)$ matrix $U = (V_+^\alpha, V_-^\alpha)$, $\alpha = 1, 2$ such that

$$V_-^\alpha V_+^\beta - V_-^\beta V_+^\alpha = \epsilon^{\alpha\beta} , \quad (2.1)$$

where $\epsilon^{12} = 1 = \epsilon_{12}$, $(V_-^1)^* = V_+^2$ and $(V_-^2)^* = V_+^1$. The signs denote $U(1) \subset SU(1, 1)$ charge. Then

$$\begin{aligned} P_M &= -\epsilon_{\alpha\beta} V_+^\alpha \partial_M V_+^\beta \\ Q_M &= -i\epsilon_{\alpha\beta} V_-^\alpha \partial_M V_+^\beta . \end{aligned} \quad (2.2)$$

The three-form field strengths $G_{MNR}^\alpha = 3\partial_{[M} A_{NR]}^\alpha$, with $(A_{MN}^1)^* = A_{MN}^2$ combine into the complex field strength

$$G_{MNR} = -\epsilon_{\alpha\beta} V_+^\alpha G_{MNR}^\beta . \quad (2.3)$$

The five-form self-dual field strength is

$$F_{M_1 M_2 M_3 M_4 M_5} = 5\partial_{[M_1} A_{M_2 M_3 M_4 M_5]} + \frac{5i}{8}\epsilon_{\alpha\beta} A_{[M_1 M_2}^\alpha G_{M_3 M_4 M_5]}^\beta , \quad (2.4)$$

⁴For a recent account of the description of the field strengths in terms of gauge potentials see [19].

where $F_{M_1\dots M_5} = -\frac{1}{5!}\epsilon_{M_1\dots M_5}{}^{N_1\dots N_5}F_{N_1\dots N_5}$ and $\epsilon_{01\dots 9} = 1$. To identify the scalars define the variables $\rho = V_-^2/V_-^1$ and

$$\rho = \frac{1+i\tau}{1-i\tau} . \quad (2.5)$$

In turn $\tau = \sigma + ie^{-\phi}$, where σ is the axion (RR scalar) and ϕ is the dilaton.

The Killing spinor equations of IIB supergravity are the parallel transport equations of the supercovariant derivative \mathcal{D}

$$\mathcal{D}_M \epsilon = \tilde{\nabla}_M \epsilon + \frac{i}{48} \Gamma^{N_1\dots N_4} \epsilon F_{N_1\dots N_4 M} - \frac{1}{96} (\Gamma_M{}^{N_1 N_2 N_3} G_{N_1 N_2 N_3} - 9 \Gamma^{N_1 N_2} G_{M N_1 N_2}) (C\epsilon)^* = 0 \quad (2.6)$$

and the algebraic condition

$$\mathcal{A}\epsilon = P_M \Gamma^M (C\epsilon)^* + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0 , \quad (2.7)$$

where

$$\tilde{\nabla}_M = D_M + \frac{1}{4} \Omega_{M,AB} \Gamma^{AB} , \quad D_M = \partial_M - \frac{i}{2} Q_M$$

is the spin connection, $\nabla_M = \partial_M + \frac{1}{4} \Omega_{M,AB} \Gamma^{AB}$, twisted with $U(1)$ connection Q_M , $Q_M^* = Q_M$, ϵ is a (complex) Weyl spinor, $\Gamma^{0\dots 9} \epsilon = -\epsilon$, and C is a charge conjugation matrix. For our spinor conventions⁵ see appendix A. The Killing spinor equations are the vanishing conditions of the supersymmetry transformations of the gravitino, and the supersymmetric partners of the dilaton and axion restricted to the bosonic sector of IIB supergravity, respectively. For a superspace formulation of IIB supergravity see [18]. The recent modification of IIB supergravity with ten-form potentials [19] does not change our analysis below because the Killing spinor equations remain the same.

2.2 Integrability conditions

To determine the field equations which are implied by the Killing spinor equations, one has to investigate the integrability conditions of the Killing spinor equations. This calculation is essentially the same as that of [5] where the field equations of the IIB supergravity were found from the commutator of the supersymmetry transformations. However, we cast the results in such a way that is more suitable to our purpose. The integrability conditions are

$$[\mathcal{D}_A, \mathcal{D}_B] \epsilon = \mathcal{R}_{AB} \epsilon = 0 , \quad (2.8)$$

and

$$[\mathcal{D}_A, \mathcal{A}] \epsilon = 0 , \quad (2.9)$$

where \mathcal{R} has been given in [20] and so the expression will not be repeated here. It turns out that some field equations and Bianchi identities of IIB supergravity are contained

⁵We use a mostly plus convention for the metric. To relate this to the conventions of [5], one takes $\Gamma^A \rightarrow i\Gamma^A$ and every time an index is lowered there is also an additional minus sign.

in the $\mathcal{I}_A = \frac{1}{2}\Gamma_A^{BC}\mathcal{R}_{BC}$ and $\mathcal{I} = \Gamma^A[\mathcal{D}_A, \mathcal{A}]$ components of the integrability conditions⁶. In particular, we have⁷

$$\mathcal{I}_A\epsilon = \left[\frac{1}{2}\Gamma^B E_{AB} - i\Gamma^{B_1 B_2 B_3} L F_{AB_1 B_2 B_3}\right]\epsilon - \left[\Gamma^B L G_{AB} - \Gamma_A^{B_1 \dots B_4} B G_{B_1 \dots B_4}\right](C\epsilon)^* \quad (2.10)$$

and similarly, $\Gamma^A[\mathcal{D}_A, \mathcal{A}]\epsilon$ can be written as

$$\mathcal{I}\epsilon = \left[\frac{1}{2}\Gamma^{AB} L G_{AB} + \Gamma^{A_1 \dots A_4} B G_{A_1 \dots A_4}\right]\epsilon + \left[LP + \Gamma^{AB} B P_{AB}\right](C\epsilon)^* \quad (2.11)$$

where

$$\begin{aligned} E_{AB} &:= R_{AB} - \frac{1}{2}g_{AB}R - \frac{1}{6}F_{AC_1 \dots C_4} F_B^{C_1 \dots C_4} - \frac{1}{4}G_{(A}^{C_1 C_2} G_{B)C_1 C_2}^* \\ &\quad + \frac{1}{24}g_{AB}G^{C_1 C_2 C_3} G_{C_1 C_2 C_3}^* - 2P_{(A} P_{B)}^* + g_{AB}P^C P_C^*, \\ LG_{AB} &:= \frac{1}{4}(\tilde{\nabla}^C G_{ABC} - P^C G_{ABC}^* + \frac{2i}{3}F_{ABC_1 C_2 C_3} G^{C_1 C_2 C_3}), \\ LP &:= \tilde{\nabla}^A P_A + \frac{1}{24}G_{A_1 A_2 A_3} G^{A_1 A_2 A_3}, \\ LF_{A_1 \dots A_4} &:= \frac{1}{3!}(\nabla^B F_{A_1 \dots A_4 B} + \frac{i}{288}\epsilon_{A_1 \dots A_4}^{B_1 \dots B_6} G_{B_1 B_2 B_3} G_{B_4 B_5 B_6}^*), \\ BF_{A_1 \dots A_6} &:= \frac{1}{5!}(\partial_{[A_1} F_{A_2 \dots A_6]} - \frac{5i}{12}G_{[A_1 A_2 A_3} G_{A_4 A_5 A_6]}^*), \\ BG_{A_1 \dots A_4} &:= \frac{1}{4!}(D_{[A_1} G_{A_2 A_3 A_4]} + P_{[A_1} G_{A_2 A_3 A_4]}^*), \\ BP_{AB} &:= D_{[A} P_{B]}. \end{aligned} \quad (2.12)$$

One can show that LF and BF are not independent but are related by the self duality condition on F . The field strengths P and G have different $U(1) \subset SU(1, 1)$ charges. In particular, one has

$$\begin{aligned} D_A P_B &= \partial_A P_B - 2iQ_A P_B \\ D_A G_{BCD} &= \partial_A G_{BCD} - iQ_A G_{BCD}. \end{aligned} \quad (2.13)$$

To derive the above expressions some very painful Dirac algebra is required but we have been assisted by GAMMA [22] to perform most of the computation. The algebraic Killing spinor equation (2.7) has also been used to convert expressions containing G and P fluxes. The above choice of components of integrability conditions that contain the field equations and the Bianchi identities is not unique. For example, the component $\Gamma^B \mathcal{R}_{AB}$ may also be used giving identical results. However, it turns out that the computation is more involved.

3 The five types of spinors

3.1 General case

The spinors that appear in type IIB supergravity are complex Weyl spinors of positive chirality. A direct consequence of this is that the most general Killing spinor of IIB

⁶The integrability conditions of the Killing spinor equations (2.8) and (2.9) may impose further conditions on the Bianchi identities and field equations than those implied by the vanishing of these two components.

⁷In terms of BF , we have

$$\mathcal{I}_A\epsilon = \left[\frac{1}{2}\Gamma^B E_{AB} + i\Gamma_A^{B_1 \dots B_6} B F_{B_1 \dots B_6}\right]\epsilon - \left[\Gamma^B L G_{AB} - \Gamma_A^{B_1 \dots B_4} B G_{B_1 \dots B_4}\right](C\epsilon)^*.$$

supergravity can be written as

$$\epsilon = p1 + qe_{1234} + u^i e_{i5} + \frac{1}{2}v^{ij}e_{ij} + \frac{1}{6}w^{ijk}e_{ijk5} , \quad (3.1)$$

where p, q, u, v and w are complex functions on the spacetime, and $i, j, k = 1, 2, 3, 4$. The supercovariant derivative acting on ϵ gives

$$\begin{aligned} \mathcal{D}_A \epsilon &= \partial_A p 1 + \partial_A q e_{1234} + \partial_A u^i e_{i5} + \frac{1}{2} \partial_A v^{ij} e_{ij} + \frac{1}{6} \partial_A w^{ijk} e_{ijk5} \\ &+ p_0 \mathcal{D}_A 1 + q_0 \mathcal{D}_A e_{1234} + u_0^i \mathcal{D}_A e_{i5} + \frac{1}{2} v_0^{ij} \mathcal{D}_A e_{ij} + \frac{1}{6} w_0^{ijk} \mathcal{D}_A e_{ijk5} \\ &+ p_1 \mathcal{D}_A (i1) + q_1 \mathcal{D}_A (ie_{1234}) + u_1^i \mathcal{D}_A (ie_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{D}_A (ie_{ij}) + \frac{1}{6} w_1^{ijk} \mathcal{D}_A (ie_{ijk5}) \end{aligned} \quad (3.2)$$

and the algebraic Killing spinor equation becomes

$$\begin{aligned} \mathcal{A} \epsilon &= p_0 \mathcal{A} 1 + q_0 \mathcal{A} e_{1234} + u_0^i \mathcal{A} e_{i5} + \frac{1}{2} v_0^{ij} \mathcal{A} e_{ij} + \frac{1}{6} w_0^{ijk} \mathcal{A} e_{ijk5} \\ &+ p_1 \mathcal{A} (i1) + q_1 \mathcal{A} (ie_{1234}) + u_1^i \mathcal{A} (ie_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{A} (ie_{ij}) + \frac{1}{6} w_1^{ijk} \mathcal{A} (ie_{ijk5}) , \end{aligned} \quad (3.3)$$

where $p = p_0 + ip_1$ and similarly for the rest of the components. Therefore to compute the Killing spinor equations of the most general spinor in IIB supergravity, it suffices to compute the supercovariant derivative and \mathcal{A} on the ten types of spinors 1, e_{1234} , e_{i5} , e_{ij} and e_{ijk5} , and $i1$, ie_{1234} , ie_{i5} , ie_{ij} and ie_{ijk5} . However it is straightforward to see that $\mathcal{D}_A (i1)$ and $\mathcal{A} (i1)$ can be easily read from the expressions of $\mathcal{D}_A 1$ and $\mathcal{A} 1$, respectively, and similarly for the rest of the pairs. The only effect is a sign which appears in those terms of the Killing spinor equation which contain the charge conjugation matrix. Of course the Killing spinor equations acting on 1 should be in addition multiplied by the complex unit i to recover those acting on $i1$, and similarly for the rest of the pairs. Therefore to construct the linear system associated with any number of Killing spinors, it suffices to compute

$$\begin{aligned} &\mathcal{D}_A 1 , \quad \mathcal{D}_A e_{1234} , \quad \mathcal{D}_A e_{i5} , \quad \mathcal{D}_A e_{ij} , \quad \mathcal{D}_A e_{ijk5} , \\ &\mathcal{A} 1 , \quad \mathcal{A} e_{1234} , \quad \mathcal{A} e_{i5} , \quad \mathcal{A} e_{ij} , \quad \mathcal{A} e_{ijk5} , \end{aligned} \quad (3.4)$$

i.e. the Killing spinor equations on five types of spinors.

It remains to show that the integrability conditions $\mathcal{I}_A, \mathcal{I}$ on a Killing spinor ϵ can also be determined in terms of those on the above five types of spinors. Since these integrability conditions are algebraic, one finds that

$$\begin{aligned} \mathcal{I}_A \epsilon &= p_0 \mathcal{I}_A 1 + q_0 \mathcal{I}_A e_{1234} + u_0^i \mathcal{I}_A e_{i5} + \frac{1}{2} v_0^{ij} \mathcal{I}_A e_{ij} + \frac{1}{6} w_0^{ijk} \mathcal{I}_A e_{ijk5} \\ &+ p_1 \mathcal{I}_A (i1) + q_1 \mathcal{I}_A (ie_{1234}) + u_1^i \mathcal{I}_A (ie_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{I}_A (ie_{ij}) + \frac{1}{6} w_1^{ijk} \mathcal{I}_A (ie_{ijk5}) \end{aligned} \quad (3.5)$$

and similarly for the \mathcal{I} integrability condition. Because the expressions for $\mathcal{I}_A (i1)$ can be easily recovered from that of $\mathcal{I}_A 1$, and similarly for the rest, one has to compute

$$\mathcal{I}_A 1 , \quad \mathcal{I}_A e_{1234} , \quad \mathcal{I}_A e_{i5} , \quad \mathcal{I}_A e_{ij} , \quad \mathcal{I}_A e_{ijk5} ,$$

$$\mathcal{I}1, \quad \mathcal{I}e_{1234}, \quad \mathcal{I}e_{i5}, \quad \mathcal{I}e_{ij}, \quad \mathcal{I}e_{ijk5}, \quad (3.6)$$

i.e. the integrability conditions on five types of spinors. In what follows, we shall give the general formulae of the Killing spinor equations and their integrability conditions acting on all five types of spinors. In the appendices, we shall list the various components of these equations in the basis (A.7).

3.2 Invariant spinors

3.2.1 Maximally supersymmetric H -backgrounds

In many cases of interest, the Killing spinors are invariant under some proper subgroup H of $Spin(9, 1)$. In such cases, it has been shown in [13] that the space of H -invariant spinors, Δ^H , is even-dimensional, $\dim \Delta^H = k = 2m$, and there is a basis $(\eta_i, i = 1, \dots, k) = (\eta_p = \eta_p, \eta_{m+p} = i\eta_p, p = 1, \dots, m)$, where η_p are H -invariant Majorana-Weyl spinors. The most general H -invariant Killing spinors in this case are

$$\epsilon_r = \sum_{i=1}^k f_{ri} \eta_i, \quad r = 1, \dots, N \quad (3.7)$$

where (f_{ri}) is a $N \times k$ matrix of real functions and N is the number of Killing spinors of the background. It has also been shown in [13] that the Killing spinor equations of backgrounds with H -invariant Killing spinors whose number of Killing spinors is equal to the real dimension of Δ^H , i.e. of maximally supersymmetric H -backgrounds, dramatically simplify. In particular the terms in Killing spinor equations that contain the P and G fluxes factorize from those that contain the F fluxes and geometry. This was shown in some special cases, here we shall present the proof of the general case.

In the maximally-supersymmetric H -backgrounds, $f = (f_{ri})$ is invertible. Because of this, the Killing spinor equations can be written as

$$\sum_{j=1}^N (f^{-1} D_M f)_{ij} \eta_j + \mathcal{D}_M \eta_i = 0, \quad \mathcal{A} \eta_i = 0. \quad (3.8)$$

First consider the algebraic Killing spinor equation for $i = 1$ and $i = m + 1$. In this case $\eta_{m+1} = i\eta_1$ and so

$$\begin{aligned} \mathcal{A} \eta_1 &= P_A \Gamma^A \eta_1 + \frac{1}{24} \Gamma^{ABC} G_{ABC} \eta_1 = 0, \\ \mathcal{A} \eta_{m+1} &= -i P_A \Gamma^A \eta_1 + \frac{i}{24} \Gamma^{ABC} G_{ABC} \eta_1 = 0. \end{aligned} \quad (3.9)$$

Therefore, we conclude that $P_A \Gamma^A \eta_1 = 0$ and $\Gamma^{ABC} G_{ABC} \eta_1 = 0$. Applying this for all pairs, we get

$$\begin{aligned} P_A \Gamma^A \eta_p &= 0, \quad p = 1, \dots, m \\ \Gamma^{ABC} G_{ABC} \eta_p &= 0, \quad p = 1, \dots, m. \end{aligned} \quad (3.10)$$

Similarly, for $i = p$ and $i = m + p$ in the first equation in (3.8), we find

$$\begin{aligned}
& \sum_{j=1}^N (f^{-1} D_M f)_{pj} \eta_j + \nabla_M \eta_p + \frac{i}{48} \Gamma^{N_1 \dots N_4} \eta_p F_{N_1 \dots N_4 M} \\
& - \frac{1}{96} (\Gamma_M^{N_1 N_2 N_3} G_{N_1 N_2 N_3} - 9 \Gamma^{N_1 N_2} G_{M N_1 N_2}) \eta_p = 0 , \\
& -i \sum_{j=1}^N (f^{-1} D_M f)_{m+pj} \eta_j + \nabla_M \eta_p + \frac{i}{48} \Gamma^{N_1 \dots N_4} \eta_p F_{N_1 \dots N_4 M} \\
& + \frac{1}{96} (\Gamma_M^{N_1 N_2 N_3} G_{N_1 N_2 N_3} - 9 \Gamma^{N_1 N_2} G_{M N_1 N_2}) \eta_p = 0 .
\end{aligned} \tag{3.11}$$

Adding and subtracting these equations, we get

$$\begin{aligned}
& \frac{1}{2} \left[\sum_{j=1}^N (f^{-1} D_M f)_{pj} \eta_j - i \sum_{j=1}^N (f^{-1} D_M f)_{m+pj} \eta_j \right] + \nabla_M \eta_p + \frac{i}{48} \Gamma^{N_1 \dots N_4} \eta_p F_{N_1 \dots N_4 M} = 0 \\
& \sum_{j=1}^N (f^{-1} D_M f)_{pj} \eta_j + i \sum_{j=1}^N (f^{-1} D_M f)_{m+pj} \eta_j + \frac{1}{4} G_{MBC} \Gamma^{BC} \eta_p = 0
\end{aligned} \tag{3.12}$$

where we have also used the second equation in (3.10). It is easy to see from (3.10) and (3.12) that, as we have mentioned, the Killing spinor equations factorize.

It has been observed in [13] that the solution to the Killing spinor equation in this case gives rise to a parallel transport equation

$$f^{-1} df + C = 0 . \tag{3.13}$$

The connection C can be thought of as the restriction of the supercovariant connection on the bundle of Killing spinors

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\mathcal{K} \rightarrow 0 , \tag{3.14}$$

where \mathcal{S} is the spin bundle of the theory. A necessary condition for the existence of a solution to the parallel transport equation is the vanishing of the curvature $F(C) = dC - C \wedge C = 0$. It is worth pointing out that for maximally supersymmetric backgrounds, $H = 1$, $\mathcal{K} = \mathcal{S}$ and C is the supercovariant connection. The curvature $F(C)$ is the supercovariant curvature $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$. The vanishing of \mathcal{R} was precisely the condition analyzed in [10] to classify the supersymmetric solutions of ten- and eleven-dimensional supergravities.

The integrability conditions of the Killing spinor equations of maximally supersymmetric H -backgrounds factorize as well. In particular since $\det f \neq 0$, it is easy to see that $\mathcal{I}_A \epsilon_i = 0$ and $\mathcal{I} \epsilon_i = 0$ imply that $\mathcal{I}_A \eta_i = 0$ and $\mathcal{I} \eta_i = 0$, $i = 1, \dots, 2m$. In turn these two equations give

$$\begin{aligned}
\left[\frac{1}{2} \Gamma^B E_{AB} - i \Gamma^{B_1 B_2 B_3} L F_{AB_1 B_2 B_3} \right] \eta_j &= 0 , \\
\left[\Gamma^B L G_{AB} - \Gamma_A^{B_1 \dots B_4} B G_{B_1 \dots B_4} \right] \eta_j &= 0 , \\
\left[\frac{1}{2} \Gamma^{AB} L G_{AB} + \Gamma^{A_1 \dots A_4} B G_{A_1 \dots A_4} \right] \eta_j &= 0 ,
\end{aligned}$$

$$[LP + \Gamma^{AB}BP_{AB}]\eta_j = 0, \quad j = 1, \dots, m. \quad (3.15)$$

It is clear that if one assumes that the Bianchi identities are satisfied, then the above conditions impose strong conditions on the field equations. We analyze these in detail for the maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ - and $Spin(7) \ltimes \mathbb{R}^8$ - backgrounds.

3.2.2 Half-maximally supersymmetric H -backgrounds

Apart from the maximally supersymmetric backgrounds above, there is also another class of backgrounds with H -invariant spinors for which the the Killing spinor equations simplify. These are the backgrounds for which the number of Killing spinors is $N = \frac{1}{2}\dim\Delta^H = m$. We refer to such backgrounds as half-maximal H -backgrounds. It turns out that if the Killing spinors are generic⁸, then the Killing spinor equations of half-maximal H backgrounds simplify but they do not necessarily factorize as in the maximal case.

The Killing spinors for half-maximal H -backgrounds can be written as

$$\epsilon_i = \sum_{j=1}^m z_{ij}\eta_j, \quad i, j = 1, \dots, m, \quad (3.16)$$

where $z = (z_{ij})$ is an $m \times m$ matrix of complex functions on the spacetime. This can be easily seen by expressing m H -invariant spinors in the basis $(\eta_j, i\eta_j)$ of Δ^H .

Next suppose that ϵ_i are generic, i.e. that $\det z \neq 0$. In such case, the algebraic Killing spinor equation can be written as

$$P_A\Gamma^A z^*\eta + \frac{1}{24}G_{ABC}\Gamma^{ABC}z\eta = 0, \quad (3.17)$$

where we have used matrix notation for z and η . This can be then rewritten as

$$P_A\Gamma^A z^{-1}z^*\eta + \frac{1}{24}G_{ABC}\Gamma^{ABC}\eta = 0. \quad (3.18)$$

Acting on this equation with Γ_A , we find that

$$(P^B\Gamma_{AB} + P_A)z^{-1}z^*\eta + \frac{1}{24}G^{BCD}\Gamma_{ABCD}\eta + \frac{1}{8}G_{ABC}\Gamma^{BC}\eta = 0. \quad (3.19)$$

Solving for the fourth order term in the gamma matrices and substituting the result into (2.6), we get

$$D_A z\eta + z\nabla_A\eta + \frac{i}{48}\Gamma^{B_1\dots B_4}z\eta F_{B_1\dots B_4 A} + \frac{1}{4}z^*z^{-1}z^*[P^B\Gamma_{AB} + P_A]\eta + \frac{1}{8}G_{ABC}\Gamma^{BC}z^*\eta = 0 \quad (3.20)$$

or equivalently,

$$z^{-1}D_A z\eta + \nabla_A\eta + \frac{i}{48}\Gamma^{B_1\dots B_4}\eta F_{B_1\dots B_4 A} + \frac{1}{4}z^{-1}z^*z^{-1}z^*[P^B\Gamma_{AB} + P_A]\eta$$

⁸We shall make this precise later.

$$+\frac{1}{8}G_{ABC}\Gamma^{BC}z^{-1}z^*\eta=0. \quad (3.21)$$

There is no factorization of the Killing spinor equations in this case, in contrast to the maximal supersymmetric H -backgrounds. Nevertheless, the Killing spinor equations simplify because the contribution of the G and the P fluxes in the (2.6) is contained in the up to gamma square terms and the F flux term is independent from the spacetime functions z . Therefore the effect of the G and P fluxes is to modify the spin connection Ω and the $U(1)$ connection Q with terms that depend on the P and G fluxes and the functions that determine the Killing spinors.

The solution to the Killing spinor equations of generic half-maximally supersymmetric H -backgrounds gives rise to a parallel transport equation

$$z^{-1}dz + C = 0. \quad (3.22)$$

The connection C can again be thought of as the restriction of the supercovariant connection on the bundle of Killing spinors \mathcal{K} . However unlike the case of maximally supersymmetric H -backgrounds, C depends on z , $C = C(z)$. To see this observe that some fluxes in (3.21) depend on the functions z . We have also confirmed this in an example. Because of this, although it is always possible to solve the linear system associated with the Killing spinor equations, the resulting parallel transport equation may be rather involved.

Next let us consider the special or degenerate cases $\det z = 0$. These cases arise whenever the Killing spinors ϵ_i are linearly dependent over the complex numbers but linearly independent over the real numbers. These cases are characterized by the rank of z . If the rank of z is $m - 1$, then it can be arranged such that the first $m - 1$ Killing spinors are linearly independent over the complex numbers but the last one is linearly dependent. In such a case, we can write

$$\epsilon_m = w_1\epsilon_1 + \dots + w_{m-1}\epsilon_{m-1}, \quad (3.23)$$

where at least one of w_1, \dots, w_{m-1} has a non-vanishing imaginary part. If all the imaginary parts vanish, then ϵ_m is linearly dependent on $\epsilon_1, \dots, \epsilon_{m-1}$ over the reals and the background has $m - 1$ supersymmetries. One can modify the above argument in the cases for which z has rank $r < m$ for $r = 1, \dots, m - 1$. It appears that the solution of the Killing spinor equations in the degenerate cases requires information on the solutions of the Killing spinor equations for $N < m$ H -invariant Killing spinors. We shall see that this is the case in the special case of $(N = 2)$ half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds.

The integrability conditions $\mathcal{I}_A\epsilon_i = \mathcal{I}\epsilon_i = 0$ also simplify for half-maximally H -supersymmetric backgrounds. We shall focus on the case where $\det z \neq 0$. In this case, these integrability conditions can be rewritten as

$$\begin{aligned} & \left[\frac{1}{2}\Gamma^B E_{AB} - i\Gamma^{B_1\dots B_3} L F_{AB_1\dots B_3} \right] \eta - \left[\Gamma^B L G_{AB} - \Gamma_A^{B_1\dots B_4} B G_{B_1\dots B_4} \right] z^{-1} z^* \eta = 0, \\ & \left[\frac{1}{2}\Gamma^{AB} L G_{AB} + \Gamma^{A_1\dots A_4} B G_{A_1\dots A_4} \right] \eta + \left[L P + \Gamma^{AB} B P_{AB} \right] z^{-1} z^* \eta = 0. \end{aligned} \quad (3.24)$$

It is clear that unlike the maximal H -backgrounds, the integrability conditions do not factorize in this case.

Combining the maximal and half-maximal cases we have mentioned above, and considering those $H \subset Spin(9, 1)$ that contain a Berger type of group, one can investigate several cases of supersymmetric backgrounds. These include backgrounds with four, eight and sixteen supersymmetries. These cases are summarized in the conclusions in table 1.

4 Linear systems

4.1 The linear system of Killing spinor equations

We have shown that the Killing spinor equations of an arbitrary spinor can be expressed in terms of those on five types of spinors given by

$$\sigma_I = e_{i_1 \dots i_I} = \frac{1}{\sqrt{2^I}} \Gamma^{\bar{i}_1 \dots \bar{i}_I} 1, \quad (4.1)$$

where the index $i = (1, \dots, 5)$ contains four holomorphic and one null indices⁹. Note that Γ^a acts as an annihilation or a creation operator on the above spinor, depending on whether the label $i_1 \dots i_I$ does or does not contain a . For this reason it is convenient to reshuffle $(\alpha, 5)$ and $(\bar{\alpha}, \bar{5})$ into indices p, q, r defined by

$$p = (\bar{i}_1, \dots, \bar{i}_I, i_{I+1}, \dots, i_5), \quad \bar{p} = (i_1, \dots, i_I, \bar{i}_{I+1}, \dots, \bar{i}_5), \quad (4.2)$$

where the indices (i_1, \dots, i_5) are some permutation of $(1, \dots, 5)$, i.e. $\epsilon_{i_1 \dots i_5} = \pm 1$. Thus, Γ^p act as annihilation operators on this spinor while $\Gamma^{\bar{p}}$ are the creation operators.

With this notation at hand, the expression for the algebraic Killing spinor equation (2.7) on an arbitrary spinor¹⁰ can be written as

$$\begin{aligned} \mathcal{A}e_{i_1 \dots i_I} &= [\tfrac{1}{4} G_{\bar{q}r}{}^r] \Gamma^{\bar{q}} e_{i_1 \dots i_I} + \\ &[\tfrac{1}{24} G_{\bar{q}_1 \dots \bar{q}_3} - \tfrac{s}{12} \epsilon_{\bar{q}_1 \dots \bar{q}_3}{}^r P_r] \Gamma^{\bar{q}_1 \dots \bar{q}_3} e_{i_1 \dots i_I} + \\ &[\tfrac{s}{96} \epsilon_{\bar{q}_1 \dots \bar{q}_4} P_{\bar{q}_5}] \Gamma^{\bar{q}_1 \dots \bar{q}_5} e_{i_1 \dots i_I}, \end{aligned} \quad (4.3)$$

where the Levi-Civita symbols are defined by $\epsilon_{\bar{1}\bar{2}\bar{3}\bar{4}} = +1$ and $\epsilon_{\bar{q}_1 \dots \bar{q}_3 \bar{5}} = 0$, i.e. these are only non-zero when all its four indices are (anti-)holomorphic. The sign s depends on whether there is a null index in $e_{i_1 \dots i_I}$:

$$s = \begin{cases} +1, & \text{for } e_{\alpha_1 \dots \alpha_I}, \\ -1, & \text{for } e_{\alpha_1 \dots \alpha_{I-1} 5}. \end{cases} \quad (4.4)$$

Observe that

$$\{e_{i_1 \dots i_I}, \Gamma^{\bar{p}} e_{i_1 \dots i_I}, \dots, \Gamma^{\bar{p}_1 \dots \bar{p}_5} e_{i_1 \dots i_I}\} \quad (4.5)$$

⁹In this section we will not use the notation $(-, +)$ for the null indices but rather $(5, \bar{5})$. Thus $\Gamma^{\bar{5}} = \Gamma^+$ and $\Gamma^5 = \Gamma^-$.

¹⁰We have used, however, that I is even for all IIB spinors.

is a basis in the space of Dirac spinors and so $e_{i_1 \dots i_I}$ can be thought of as another Clifford vacuum. Therefore the terms in the square brackets in (4.3) are linearly independent.

We can apply the same procedure to the parallel transport Killing spinor equation (2.6). In particular, we find for $M = p$ that

$$\begin{aligned} \mathcal{D}_p e_{i_1 \dots i_I} = & [D_p + \frac{1}{2} \Omega_{p,r}{}^r + \frac{i}{4} F_{p r_1}{}^{r_1}{}_{r_2}{}^{r_2}] e_{i_1 \dots i_I} + \\ & [\frac{1}{4} \Omega_{p, \bar{q}_1 \bar{q}_2} + \frac{i}{4} F_{p \bar{q}_1 \bar{q}_2}{}^r - \frac{s}{48} g_{p \bar{q}_1} \epsilon_{\bar{q}_2}{}^{r_1 \dots r_3} G_{r_1 \dots r_3} - \frac{s}{16} \epsilon_{\bar{q}_1 \bar{q}_2}{}^{r_1 r_2} G_{p r_1 r_2}] \Gamma^{\bar{q}_1 \bar{q}_2} e_{i_1 \dots i_I} + \\ & [\frac{i}{48} F_{p \bar{q}_1 \dots \bar{q}_4} + \frac{s}{96} \epsilon_{\bar{q}_1 \dots \bar{q}_3}{}^r G_{p \bar{q}_4 r} - \frac{s}{192} g_{p \bar{q}_1} \epsilon_{\bar{q}_2 \dots \bar{q}_4}{}^{r_1} G_{r_1 r_2}{}^{r_2}] \Gamma^{\bar{q}_1 \dots \bar{q}_4} e_{i_1 \dots i_I}. \end{aligned} \quad (4.6)$$

Similarly, the resulting expression for (2.6) with $M = \bar{p}$ is

$$\begin{aligned} \mathcal{D}_{\bar{p}} e_{i_1 \dots i_I} = & [D_{\bar{p}} + \frac{1}{2} \Omega_{\bar{p},r}{}^r + \frac{i}{4} F_{\bar{p} r_1}{}^{r_1}{}_{r_2}{}^{r_2} - \frac{s}{24} \epsilon_{\bar{p}}{}^{r_1 \dots r_3} G_{r_1 \dots r_3}] e_{i_1 \dots i_I} + \\ & [\frac{1}{4} \Omega_{\bar{p}, \bar{q}_1 \bar{q}_2} + \frac{i}{4} F_{\bar{p} \bar{q}_1 \bar{q}_2}{}^r + \frac{s}{32} \epsilon_{\bar{p} \bar{q}_1 \bar{q}_2}{}^{r_1} G_{r_1 r_2}{}^{r_2} \\ & - \frac{s}{32} \epsilon_{\bar{q}_1 \bar{q}_2}{}^{r_1 r_2} G_{\bar{p} r_1 r_2} + \frac{s}{16} \epsilon_{\bar{p} \bar{q}_1}{}^{r_1 r_2} G_{\bar{q}_2 r_1 r_2}] \Gamma^{\bar{q}_1 \bar{q}_2} e_{i_1 \dots i_I} + \\ & [\frac{i}{48} F_{\bar{p} \bar{q}_1 \dots \bar{q}_4} + \frac{s}{256} \epsilon_{\bar{q}_1 \dots \bar{q}_4} G_{\bar{p} r}{}^r + \frac{s}{192} \epsilon_{\bar{p} \bar{q}_1 \dots \bar{q}_3} G_{\bar{q}_4 r}{}^r \\ & + \frac{s}{48} \epsilon_{\bar{q}_1 \dots \bar{q}_3}{}^r G_{\bar{p} \bar{q}_4 r}] \Gamma^{\bar{q}_1 \dots \bar{q}_4} e_{i_1 \dots i_I}. \end{aligned} \quad (4.7)$$

To derive the linear system associated with the Killing spinor equations of a background with any number of supersymmetries, (4.3), (4.6) and (4.7) must be converted from the oscillator basis (4.5) to the ‘‘canonical basis’’

$$\{1, \Gamma^{\bar{i}} 1, \dots, \Gamma^{\bar{i}_1 \dots \bar{i}_5} 1\}. \quad (4.8)$$

To achieve this, we expand the products of $\Gamma^{\bar{p}}$ matrices, which are creation operators on $e_{i_1 \dots i_I}$, into a sum of products of $\Gamma^{\bar{j}}$ and $\Gamma^{\bar{j}}$ matrices, which are annihilation and creation operators, respectively, on 1. Then we act on $e_{i_1 \dots i_I}$ with the annihilation operators. In particular, we have

$$\begin{aligned} \mathcal{A} e_{i_1 \dots i_I} &= \sum_l [\mathcal{A} e_{i_1 \dots i_I}]_{\bar{p}_1 \dots \bar{p}_l} \Gamma^{\bar{p}_1 \dots \bar{p}_l} e_{i_1 \dots i_I} \\ &= \sum_l \sum_{m+n=l} \frac{l!}{m!n!} [\mathcal{A} e_{i_1 \dots i_I}]_{j_1 \dots j_m \bar{k}_1 \dots \bar{k}_n} \Gamma^{j_1 \dots j_m} \Gamma^{\bar{k}_1 \dots \bar{k}_n} e_{i_1 \dots i_I} \\ &= \sum_l \sum_{m+n=l} \frac{l!}{m!n!} \frac{(-1)^{[m/2]+nI}}{2^{I/2-m}(I-m)!} \epsilon^{j_1 \dots j_m \bar{j}_{m+1} \dots \bar{j}_I} \\ & \quad [\mathcal{A} e_{i_1 \dots i_I}]_{j_1 \dots j_m \bar{k}_1 \dots \bar{k}_n} \Gamma^{\bar{j}_{m+1} \dots \bar{j}_I \bar{k}_1 \dots \bar{k}_n} 1, \end{aligned} \quad (4.9)$$

with the obvious restrictions $m \leq I$ and $n \leq 5 - I$ and the convention that $\epsilon_{\bar{i}_1 \dots \bar{i}_I} = 1$. A similar formula holds for all components of $\mathcal{D}_{\mathcal{M}}$. Using these expressions one can easily compute the components of $\mathcal{A} e_{i_1 \dots i_I}$ and $\mathcal{D}_{M e_{i_1 \dots i_I}}$ in the canonical basis (4.8). For convenience we give the explicit expressions for $\mathcal{A} \sigma_I$ and $\mathcal{D} \sigma_I$ in the appendices.

4.2 The linear system of integrability conditions

The integrability conditions of the Killing spinor equations (2.11) and (2.10) of a IIB background with any number of supersymmetries can also be expressed in terms of those on five types of spinors σ_I . To expand $\mathcal{I}\sigma_I$ and $\mathcal{I}_A\sigma_I$ in the canonical basis (4.8), we follow the same procedure as that for the Killing spinor equations in the previous section. In particular, we first compute the integrability condition \mathcal{I} in the oscillator basis (4.5) to find

$$\begin{aligned}\mathcal{I}e_{i_1\dots i_I} &= [LG_r{}^r + 12BG_{r_1}{}^{r_1}{}_{r_2}{}^{r_2}]e_{i_1\dots i_I} + [\tfrac{1}{2}LG_{\bar{q}_1\bar{q}_2} \\ &\quad + 12BG_{\bar{q}_1\bar{q}_2}{}^r - \tfrac{s}{2}\epsilon_{\bar{q}_1\bar{q}_2}{}^{r_1r_2}BP_{r_1r_2}]\Gamma^{\bar{q}_1\bar{q}_2}e_{i_1\dots i_I} + [BG_{\bar{q}_1\dots\bar{q}_4} \\ &\quad + \tfrac{s}{96}\epsilon_{\bar{q}_1\dots\bar{q}_4}LP + \tfrac{s}{48}\epsilon_{\bar{q}_1\dots\bar{q}_4}BP_r{}^r + \tfrac{s}{6}\epsilon_{\bar{q}_1\dots\bar{q}_3}{}^rBP_{\bar{q}_4r}]\Gamma^{\bar{q}_1\dots\bar{q}_4}e_{i_1\dots i_I} .\end{aligned}\quad (4.10)$$

Similarly for the integrability condition \mathcal{I}_A , we get

$$\begin{aligned}\mathcal{I}_p e_{i_1\dots i_I} &= [\tfrac{1}{2}E_{p\bar{q}} - 6iLF_{p\bar{q}}{}^r + 4sg_{p\bar{q}}\epsilon^{r_1\dots r_4}BG_{r_1\dots r_4} - 8s\epsilon_{\bar{q}}{}^{r_1\dots r_3}BG_{pr_1\dots r_3}]\Gamma^{\bar{q}}e_{i_1\dots i_I} + \\ &\quad [-iLF_{p\bar{q}_1\dots\bar{q}_3} + \tfrac{s}{12}\epsilon_{\bar{q}_1\dots\bar{q}_3}{}^rLG_{pr} - 4sg_{p\bar{q}_1}\epsilon_{\bar{q}_2}{}^{r_1\dots r_3}BG_{\bar{q}_3r_1\dots r_3} + \\ &\quad - 2s\epsilon_{\bar{q}_1\dots\bar{q}_3}{}^{r_1}BG_{pr_1r_2}{}^{r_2}]\Gamma^{\bar{q}_1\dots\bar{q}_3}e_{i_1\dots i_I} + [-\tfrac{s}{96}\epsilon_{\bar{q}_1\dots\bar{q}_4}LG_{p\bar{q}_5} \\ &\quad + \tfrac{1}{8}sg_{p\bar{q}_1}\epsilon_{\bar{q}_2\dots\bar{q}_5}BG_{r_1}{}^{r_1}{}_{r_2}{}^{r_2} - \tfrac{1}{4}s\epsilon_{\bar{q}_1\dots\bar{q}_4}BG_{p\bar{q}_5}{}^r]\Gamma^{\bar{q}_1\dots\bar{q}_5}e_{i_1\dots i_I} ,\end{aligned}\quad (4.11)$$

and

$$\begin{aligned}\mathcal{I}_{\bar{p}} e_{i_1\dots i_I} &= [\tfrac{1}{2}E_{\bar{p}\bar{q}} - 6iLF_{\bar{p}\bar{q}}{}^r - 16s\epsilon_{\bar{p}}{}^{r_1\dots r_3}BG_{\bar{q}r_1\dots r_3} + 8s\epsilon_{\bar{q}}{}^{r_1\dots r_3}BG_{\bar{p}r_1\dots r_3} + \\ &\quad - 24s\epsilon_{\bar{p}\bar{q}}{}^{r_1r_2}BG_{r_1r_2r_3}{}^{r_3}]\Gamma^{\bar{q}}e_{i_1\dots i_I} + \\ &\quad [-iLF_{\bar{p}\bar{q}_1\dots\bar{q}_3} + \tfrac{s}{12}\epsilon_{\bar{q}_1\dots\bar{q}_3}{}^rLG_{\bar{p}r} - 9s\epsilon_{\bar{p}\bar{q}_1}{}^{r_1r_2}BG_{\bar{q}_2\bar{q}_3r_1r_2} - 3s\epsilon_{\bar{q}_1\bar{q}_2}{}^{r_1r_2}BG_{\bar{p}\bar{q}_3r_1r_2} + \\ &\quad + 6s\epsilon_{\bar{p}\bar{q}_1\bar{q}_2}{}^{r_1}BG_{\bar{q}_3r_1r_2}{}^{r_2}]\Gamma^{\bar{q}_1\dots\bar{q}_3}e_{i_1\dots i_I} + \\ &\quad [-\tfrac{s}{96}\epsilon_{\bar{q}_1\dots\bar{q}_4}LG_{\bar{p}\bar{q}_5} + \tfrac{s}{4}\epsilon_{\bar{q}_1\dots\bar{q}_4}BG_{\bar{p}\bar{q}_5}{}^r]\Gamma^{\bar{q}_1\dots\bar{q}_5}e_{i_1\dots i_I} .\end{aligned}\quad (4.12)$$

It remains to convert the above expressions from the oscillator basis (4.5) to the canonical basis (4.8). This can be done as in (4.9) and we shall not repeat the formula here. The explicit expressions of the integrability conditions in the canonical basis can be found in the appendices.

5 Generic $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors

5.1 Preliminaries

Applying the results of section 3.2 to this case, one finds that the two Killing spinors of $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors can be written as

$$\epsilon = z \eta \quad (5.1)$$

where $\eta_1 = 1 + e_{1234}$ and $\eta_2 = i(1 - e_{1234})$ and z is a complex 2×2 matrix. There are two classes of such backgrounds. For *generic* half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds the matrix z is non-degenerate, $\det z \neq 0$, whereas the *degenerate* half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds have $\det z = 0$ but the two Killing spinors are linearly independent over the real numbers. To solve the Killing spinor equations in the generic case, we adapt the formalism developed in section 3.2. In this section, we shall investigate the generic class of half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. The degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds will be examined in section 6.

5.2 The solution of the linear system

The linear system and its solution are described in appendix D. It turns out that to solve the linear system for generic half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds one has to consider two cases depending on whether or not $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$, where $A = z^{-1}z^*$. In both cases after solving for the fluxes, the resulting equations can be separated into two classes. One class are the algebraic equations which do not contain spacetime derivatives of the functions z , e.g. (D.54) and (D.57). The other case are first order equations for the functions z which however are *non-linear* in z , e.g. (D.53), (D.52), (D.61) and (D.62). These first order equations can be viewed as the parallel transport equations of the restriction of the supercovariant connection on the bundle of the Killing spinors \mathcal{K} . However, since the system is non-linear, the analysis of the general case is rather involved. So instead of solving the system in general, we shall investigate two examples. In the first example $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0$ while in the second $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$.

5.2.1 Special cases

The matrix z in the example with $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0$ is chosen as

$$z = \begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix}, \quad z_{11} \neq z_{22} \neq 0, \quad (5.2)$$

where z_{11} and z_{22} are complex. Without loss of generality, we shall also work in the gauge for which $\det z = 1$ and so $z_{11}z_{22} = 1$. This gauge can always be locally attained by an appropriate $U(1)$ gauge transformation and an appropriate $Spin(9,1)$ transformation along Γ_{05} . We therefore can set

$$z_{11} = \rho_1 e^{i\varphi}, \quad z_{22} = \rho_2 e^{-i\varphi}, \quad \rho_1 \rho_2 = 1 \quad (5.3)$$

for $\rho_1 = \rho, \rho_2 = \rho^{-1}, \phi \in \mathbb{R}, \rho > 0$. Consequently, $A = \text{diag}(e^{-2i\varphi}, e^{2i\varphi})$.

We shall not go through a detailed analysis of the solution. This has been done in D.3. Instead, we shall summarize the conditions on the geometry and fluxes. In particular we find that the conditions on the geometry are

$$\begin{aligned} \Omega_{\alpha,\beta+} &= 0, & \Omega_{+,\alpha\beta} &= 0, & \Omega_{+,+\alpha} &= 0, & Q_+ &= Q_\alpha = 0, & \Omega_{+,-+} &= 0 \\ \partial_+\varphi &= 0, & \partial_A\rho &= 0, & \Omega_{+,\alpha}{}^\alpha &= 0, & \Omega_{-,-+} &= 0, & \Omega_{\alpha,\bar{\beta}+} &= 0, \end{aligned}$$

$$\begin{aligned}
\Omega_{\beta, \bar{\alpha}}^{\beta} &= -\frac{3}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} , \quad \Omega_{[\beta_1, \beta_2, \beta_3]} = 0 , \\
\Omega_{\alpha, -+} + \Omega_{-, \alpha+} &= 0 , \quad \partial_{\bar{\alpha}} \varphi = -\frac{\sin(4\varphi)}{\cos(4\varphi) + 2} \Omega_{\bar{\alpha}, -+} , \\
\Omega_{\bar{\alpha}, \beta}^{\beta} + \frac{\cos(4\varphi)}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} &= 0 , \quad \Omega_{\alpha, \bar{\beta}\bar{\gamma}} = \frac{2}{\cos(4\varphi) + 2} \Omega_{-, +[\bar{\beta}g\bar{\gamma}]\alpha} ,
\end{aligned} \tag{5.4}$$

the conditions on the G and P fluxes are

$$\begin{aligned}
G_{+\alpha}^{\alpha} &= 0 , \quad P_{+} = 0 , \quad G_{+\alpha\beta} = G_{+\bar{\alpha}\bar{\beta}} = 0 , \quad G_{-\alpha}^{\alpha} = -\frac{2}{\cos(2\varphi)} \Omega_{-, \alpha}^{\alpha} \\
G_{-\bar{\alpha}\bar{\beta}} &= -2 \cos(2\varphi) (\Omega_{-, \bar{\alpha}\bar{\beta}} + i F_{-\bar{\alpha}\bar{\beta}\gamma}^{\gamma}) + i \sin(2\varphi) \epsilon_{\bar{\alpha}\bar{\beta}}^{\gamma\delta} (\Omega_{-, \gamma\delta} - i F_{-\gamma\delta\zeta}^{\zeta}) \\
G_{-\alpha\beta} &= -2 \cos(2\varphi) (\Omega_{-, \alpha\beta} - i F_{-\alpha\beta\gamma}^{\gamma}) + i \sin(2\varphi) \epsilon_{\alpha\beta}^{\bar{\gamma}\bar{\delta}} (\Omega_{-, \bar{\gamma}\bar{\delta}} + i F_{-\bar{\gamma}\bar{\delta}\zeta}^{\zeta}) \\
P_{\bar{\alpha}} &= (P_{\alpha})^{*} = \frac{2}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} , \quad G_{-+\bar{\alpha}} = (G_{-+\alpha})^{*} = -8 \frac{\cos(2\varphi)}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} \\
G_{\bar{\alpha}\beta}^{\beta} &= G_{\alpha\beta}^{\beta} = 0 , \quad \epsilon_{\bar{\alpha}}^{\beta_1\beta_2\beta_3} G_{\beta_1\beta_2\beta_3} = -(\epsilon_{\alpha}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3})^{*} = 24i \frac{\sin(2\varphi)}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} \\
G_{+\alpha\bar{\beta}} &= 0 , \quad G_{\bar{\alpha}\beta\gamma} = -(G_{\alpha\bar{\beta}\bar{\gamma}})^{*} = i \frac{1}{\sin(2\varphi)} \Omega_{\bar{\alpha}, \bar{\delta}_1\bar{\delta}_2} \epsilon_{\beta\gamma}^{\bar{\delta}_1\bar{\delta}_2}
\end{aligned} \tag{5.5}$$

and the conditions on the F fluxes, in addition to the self-duality, are

$$\begin{aligned}
F_{+\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} &= 0 , \quad F_{-\alpha}^{\alpha\beta} = 2Q_{-} , \quad F_{-\beta_1\beta_2\beta_3\beta_4} \epsilon^{\beta_1\beta_2\beta_3\beta_4} = -12\partial_{-}\varphi - 6 \tan(2\varphi) \Omega_{-, \alpha}^{\alpha} , \\
\epsilon_{\bar{\alpha}}^{\beta_1\beta_2\beta_3} F_{-+\beta_1\beta_2\beta_3} &= -3 \frac{\sin(4\varphi)}{\cos(4\varphi) + 2} \Omega_{-, +\bar{\alpha}} , \quad F_{-+\bar{\alpha}\beta}^{\beta} = 0 , \quad F_{+\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2} = 0 \\
F_{-+\alpha\bar{\beta}\bar{\gamma}} &= -\frac{1}{4} \cotan(2\varphi) \Omega_{\alpha, \gamma_1\gamma_2} \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}\bar{\gamma}} , \\
i \text{Im}(F_{-\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}) &= -6 \tan(2\varphi) \Omega_{-, \alpha}^{\alpha} .
\end{aligned} \tag{5.6}$$

The conditions above have an explicit dependence on the angle φ . This is due to the non-linearity of the Killing spinor equation on the functions z .

The other special case that we shall consider is to take z to be a real matrix. In this case, $A = 1_{2 \times 2}$ is the identity matrix and the non-linear system becomes linear. This case closely resembles the maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ case that we have investigated in [13]. As in the previous case, we shall simply summarize the solution of the linear system. The conditions on the geometry are

$$\begin{aligned}
(z^{-1}\partial_{+}z)_{11} &= (z^{-1}\partial_{+}z)_{22} = -\frac{1}{2} \Omega_{+, -+} , \\
(z^{-1}\partial_{+}z)_{12} &= -(z^{-1}\partial_{+}z)_{21} = \frac{i}{2} \Omega_{+, \alpha}^{\alpha} , \\
(z^{-1}\partial_{-}z)_{11} &= (z^{-1}\partial_{-}z)_{22} = -\frac{1}{2} \Omega_{-, -+} , \\
(z^{-1}\partial_{-}z)_{12} &= -(z^{-1}\partial_{-}z)_{21} = \frac{i}{2} \Omega_{-, \alpha}^{\alpha} + \frac{i}{4} G_{-\alpha}^{\alpha} , \\
(z^{-1}\partial_{\bar{\alpha}}z)_{11} &= (z^{-1}\partial_{\bar{\alpha}}z)_{22} = -\frac{1}{2} \Omega_{\bar{\alpha}, -+} - \frac{1}{2} \Omega_{\beta, \bar{\alpha}}^{\beta} , \\
(z^{-1}\partial_{\bar{\alpha}}z)_{12} &= -(z^{-1}\partial_{\bar{\alpha}}z)_{21} = \frac{i}{2} \Omega_{\bar{\alpha}, \beta}^{\beta} - \frac{i}{6} \Omega_{\beta, \bar{\alpha}}^{\beta} + \frac{i}{12} (G_{\bar{\alpha}\beta}^{\beta} - (G_{\alpha\beta}^{\beta})^{*}) ,
\end{aligned}$$

$$\begin{aligned}
Q_+ = 0, \quad \Omega_{\alpha,\beta+} = 0, \quad \Omega_{+, \alpha\beta} = 0, \quad \Omega_{+, +\alpha} = 0, \\
\Omega_{\alpha, +\bar{\beta}} + \Omega_{\bar{\beta}, +\alpha} = 0, \quad \Omega_{\alpha_1, \alpha_2 \alpha_3} = 0, \quad \Omega_{-+\bar{\alpha}} + \Omega_{\beta, \bar{\alpha}}^\beta = 0,
\end{aligned} \tag{5.7}$$

the conditions on the P and G fluxes are

$$\begin{aligned}
P_+ = 0, \quad G_{+\alpha}^\alpha = 0, \quad G_{+\alpha\beta} = G_{+\bar{\alpha}\bar{\beta}} = 0, \quad G_{-\alpha}^\alpha + (G_{-\alpha}^\alpha)^* = 0, \\
G_{-\alpha\beta} = -2(\Omega_{-, \alpha\beta} - iF_{-\alpha\beta\gamma}^\gamma), \quad G_{-\bar{\alpha}\bar{\beta}} = -2(\Omega_{-\bar{\alpha}\bar{\beta}} + iF_{-\bar{\alpha}\bar{\beta}\gamma}^\gamma), \\
G_{+\alpha\bar{\beta}} = \Omega_{\alpha, +\bar{\beta}} - \Omega_{\bar{\beta}, +\alpha}, \quad P_{\bar{\alpha}} = -\frac{1}{3}G_{\bar{\alpha}\beta}^\beta - \frac{2}{3}(\Omega_{\beta, \bar{\alpha}}^\beta - iF_{-+\bar{\alpha}\beta}^\beta), \\
G_{-+\bar{\alpha}} = \frac{1}{3}G_{\bar{\alpha}\beta}^\beta + \frac{8}{3}(\Omega_{\beta, \bar{\alpha}}^\beta - iF_{-+\bar{\alpha}\beta}^\beta), \quad G_{\alpha\beta\gamma} = G_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 0, \\
P_\alpha = \frac{1}{3}G_{\alpha\beta}^\beta - \frac{2}{3}(\Omega_{\bar{\beta}, \alpha}^\beta + iF_{-+\alpha\beta}^\beta), \quad G_{-+\alpha} = -\frac{1}{3}G_{\alpha\beta}^\beta + \frac{8}{3}(\Omega_{\bar{\beta}, \alpha}^\beta + iF_{-+\alpha\beta}^\beta), \\
G_{\alpha\bar{\beta}_1\bar{\beta}_2} = -2(\Omega_{\alpha, \bar{\beta}_1\bar{\beta}_2} + 2iF_{-+\alpha\bar{\beta}_1\bar{\beta}_2}) - \delta_{\alpha[\bar{\beta}_1} \left(-\frac{2}{3}G_{\bar{\beta}_2]\gamma}^\gamma - \frac{4}{3}\Omega_{|\gamma|, \bar{\beta}_2]} - \frac{8i}{3}F_{\bar{\beta}_2]-+\gamma}^\gamma \right) \\
G_{\bar{\alpha}\beta_1\beta_2} = -2(\Omega_{\bar{\alpha}, \beta_1\beta_2} + 2iF_{-+\bar{\alpha}\beta_1\beta_2}) - \delta_{\bar{\alpha}[\beta_1} \left(\frac{2}{3}G_{\beta_2]\gamma}^\gamma - \frac{4}{3}\Omega_{|\gamma|, \bar{\beta}_2]} + \frac{8i}{3}F_{\bar{\beta}_2]-+\gamma}^\gamma \right) \tag{5.8}
\end{aligned}$$

and the conditions on the F fluxes, in addition to the self-duality, are

$$\begin{aligned}
F_{+\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0, \quad F_{-\alpha}^\alpha\beta^\beta = 2Q_-, \quad F_{-\beta_1\beta_2\beta_3\beta_4} = 0, \\
F_{+\alpha\bar{\beta}\gamma}^\gamma = 0, \quad F_{-+\alpha\beta}^\beta = \frac{1}{2}Q_\alpha, \quad F_{-+\bar{\alpha}\beta}^\beta = \frac{i}{8}(G_{\bar{\alpha}\beta}^\beta + (G_{\alpha\beta}^\beta)^*), \\
F_{-+\alpha_1\alpha_2\alpha_3} = 0.
\end{aligned} \tag{5.9}$$

Observe that the conditions have been expressed in representations of $SU(4) \ltimes \mathbb{R}^8$ as may have been expected.

5.3 The geometry

To investigate the geometry of generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ backgrounds, one has to solve the parallel transport equation

$$z^{-1}dz + C = 0. \tag{5.10}$$

However as we have seen that the connection depends on z , $C = C(z)$, the first order system is non-linear. Because of this, we shall focus on the geometry of two examples we have described in the previous section.

First, let us consider the case for which z is diagonal. The conditions on the geometry (5.4) imply that the Killing spinors can be written as

$$\epsilon_1 = e^{i\varphi}\eta_1, \quad \epsilon_2 = e^{-i\varphi}\eta_2, \tag{5.11}$$

where φ depends on the spacetime coordinates. In addition φ satisfies a parallel transport equation

$$d\varphi + C = 0 \tag{5.12}$$

where

$$C_+ = 0, \quad C_- = \frac{1}{2}\tan(2\varphi)\Omega_{-, \alpha}^\alpha + \frac{1}{12}F_{-\beta_1\beta_2\beta_3\beta_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4},$$

$$C_{\bar{a}} = \frac{\sin(4\varphi)}{\cos(4\varphi) + 2} \Omega_{\bar{a}, -+} . \quad (5.13)$$

Observe that C depends on the angle φ as expected. The dependence of ϵ_1, ϵ_2 on the angle φ cannot be eliminated with a $Spin(9, 1) \times U(1)$ gauge transformation because we have already used such transformations to simplify the Killing spinors¹¹. The angle φ is determined by the field equations.

To investigate further the geometry of these backgrounds, one can compute the space-time form bi-linears associated with the Killing spinors (5.11). This can be easily done using the results in appendix A. For this we introduce a light-cone frame and write the spacetime metric as

$$ds^2 = 2e^- e^+ + \delta_{IJ} e^I e^J , \quad I, J = 1, \dots, 8 . \quad (5.14)$$

Then after an appropriate normalization, one finds that the ring of Killing spinor bi-linears is generated by

$$\kappa = e^- , \quad \xi = e^- \wedge \omega , \quad \tau = e^- \wedge \chi , \quad \tau^* = e^- \wedge \chi^* , \quad \lambda = e^- \wedge \omega \wedge \omega \quad (5.15)$$

as can be seen from the bi-linears $\kappa(\epsilon_1, \tilde{\epsilon}_1)$, $\xi(\epsilon_1, \epsilon_2)$, $\tau(\epsilon_1, \epsilon_2)$, $\tau(\epsilon_1, \tilde{\epsilon}_1)$ and $\tau(\epsilon_2, \tilde{\epsilon}_2)$. Observe that the remaining bi-linears in appendix A depend on the angle φ . Clearly the ring of bi-linears is two step nilpotent¹². The one-form κ is associated with a *null Killing* vector field $X = e_+$. However unlike the case of maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds, X is not rotation free. It is straightforward to express the various components of the Levi-Civita connection Ω that appear in the geometry relations (5.4) in terms of the covariant derivatives of the generators (5.15). Then one could rewrite (5.4) in terms of the covariant derivatives. However, we shall not do this here. Observe that the various geometry conditions in (5.4) depend on the angle φ and that the covariant derivatives of (5.15) do not depend on φ . As a result, the geometry conditions (5.4) are not simply linear relations between the various components of the covariant derivatives of (5.15) as in the case of Hermitian manifolds in [23].

Next, let us take z to be a real invertible matrix. The parallel transport equation for z is given in (5.7). It is easy to see that the connection C can be written as

$$C = \hat{\Omega}^0 t_0 + \hat{\Omega}^1 t_1 \quad (5.16)$$

where $t_0 = 1_{2 \times 2}$, t_1 is skew-symmetric with $(t_1)_{12} = 1$, and $\hat{\Omega}^0$ and $\hat{\Omega}^1$ are easily computed from (5.7). Since t_0 and t_1 commute, C is an abelian connection. In addition C does not depend on z because $A = 1_{2 \times 2}$ in this example. A necessary condition for the existence of a solution to the parallel transport problem is that the curvature of C , $F = dC$, must vanish. In addition, it turns out that C can be trivialized with the $e^{a\Gamma_{05} + b\Gamma_{16}}$ gauge $Spin(9, 1)$ transformation for suitable choices of the parameters a, b . Therefore, we have

¹¹We do not expect additional $Spin(9, 1) \times U(1)$ gauge transformations that preserve the space spanned by ϵ_1, ϵ_2 to exist, apart from the subgroup $Spin(1, 1) \times U(1)$ that we have already used.

¹²Compare this with the ring of invariant forms on 2n-dimensional manifolds with an $SU(n)$ -structure which is not nilpotent.

shown that up to a $Spin(9, 1)$ gauge transformation, we can set $z = 1_{2 \times 2}$ and so the two Killing spinors¹³

$$\begin{aligned}\epsilon_1 &= \eta_1 \\ \epsilon_2 &= \eta_2 .\end{aligned}\tag{5.17}$$

Setting $z = 1_{2 \times 2}$ in (5.7), the resulting equations are interpreted either as restrictions on the geometry or as conditions that relate components of the fluxes to the geometry.

To further investigate the geometry, we write the spacetime metric in a light-cone frame as in (5.14). Then it can be easily seen from the results of appendix A that the ring of $SU(4) \ltimes \mathbb{R}^8$ forms is generated by the forms in (5.15). However unlike the previous case, all spinor bi-linears are constant in the frame e^-, e^+, e^I . The ring of spinor bi-linears is again two step nilpotent. It is easy to see from the conditions on the geometry that κ is a *null Killing* vector field. Unlike the maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds, κ is not rotation free. In particular observe that $(d\kappa)_{\alpha\bar{\beta}}$ is proportional to $G_{+\alpha\bar{\beta}}$. One can proceed further to re-express the conditions on the geometry in terms of the Levi-Civita covariant derivatives of the spacetime forms $\kappa, \xi, \tau, \tau^*$ and λ as suggested in [23]. However, we shall not do this here.

6 Degenerate $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors

6.1 Preliminaries

The Killing spinors of degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds can be written as

$$\begin{aligned}\epsilon_1 &= (f - g_2 + ig_1)1 + (f + g_2 + ig_1)e_{1234} \\ \epsilon_2 &= w\epsilon_1 ,\end{aligned}\quad \text{Im} w \neq 0 ,\tag{6.1}$$

where f, g_1, g_2 are real functions, $f, g_2 \neq 0$. To see this, it can always be arranged such that $z_{11} \neq 0$. If this is the case, then solving the condition $\det z = 0$ for z_{22} and substituting it into the second spinor, one finds that $\epsilon_2 = w\epsilon_1$, where $w = z_{21}/z_{11}$. If w is real, then η_1 and η_2 are linearly dependent and so this case is excluded. Thus, we take w to be complex with $\text{Im } w \neq 0$.

The algebraic Killing spinor equation for ϵ_2 can be written as

$$w^{-1}w^*P_A\Gamma^AC\epsilon_1^* + \frac{1}{24}G_{ABC}\Gamma^{ABC}\epsilon_1 = 0 .\tag{6.2}$$

Subtracting this from the algebraic Killing spinor equation of ϵ_1 and using $\text{Im } w \neq 0$, one finds that

$$P_A\Gamma^AC\epsilon_1^* = 0 , \quad G_{ABC}\Gamma^{ABC}\epsilon_1 = 0 .\tag{6.3}$$

¹³It is not always possible to find a gauge such that the Killing spinors of a supersymmetric background are all constant. This has to be shown in each case. An explanation of this has been given in [13]. As an example, it can be shown that the only maximally supersymmetric background of IIB supergravity with constant Killing spinors is locally isometric to Minkowski spacetime but it is known that there are two more maximally supersymmetric backgrounds the $AdS_5 \times S^5$ and the plane wave.

Similarly using $\mathcal{D}\epsilon_1 = 0$, the Killing spinor equations of ϵ_2 associated with the supercovariant derivative \mathcal{D} become

$$\partial_A w \epsilon_1 - \frac{1}{96}(w^* - w)[\Gamma_A^{B_1 B_2 B_3} G_{B_1 B_2 B_3} - 9\Gamma^{B_1 B_2} G_{AB_1 B_2}]C\epsilon_1^* = 0. \quad (6.4)$$

In turn the Killing spinor equation associated with the supercovariant derivative of ϵ_1 can be rewritten as

$$\tilde{\nabla}_A \epsilon_1 - (w^* - w)^{-1} \partial_A w \epsilon_1 + \frac{i}{48} \Gamma^{N_1 \dots N_4} \epsilon_1 F_{N_1 \dots N_4 M} = 0. \quad (6.5)$$

Therefore the independent equations that have to be solved in this case are (6.3), (6.4) and (6.5). It is clear that in this special case, the Killing spinor equations factorize in a way similar to that we have encountered for maximally supersymmetric H -backgrounds. The linear system associated with the above Killing spinor equations is given in appendix E.

6.2 The solution of the linear system

The linear system associated with the Killing spinor equations has been presented in appendix E. We shall not explain in detail the solution of this system. It turns out that it is simpler than that of $N = 1$ backgrounds with a $SU(4) \ltimes \mathbb{R}^8$ invariant Killing spinor [12]. First, we shall summarize the conditions that are implied from the equations involving the G and P fluxes and then we shall give the conditions that are implied by the rest of the equations. In particular, we have

$$\begin{aligned} \partial_+ w = \partial_- w = \partial_\alpha w = \partial_{\bar{\alpha}} w &= 0 \\ P_+ = G_{-\beta}{}^\beta = G_{+\beta}{}^\beta = G_{-\alpha}{}^\alpha = G_{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3} = G_{\alpha\beta}{}^\beta &= 0 \\ G_{+\alpha\beta} = G_{+\bar{\alpha}\bar{\beta}} = G_{-\alpha}{}^\alpha = G_{\alpha_1 \alpha_2 \alpha_3} = G_{\bar{\alpha}\beta}{}^\beta = G_{+\alpha\bar{\beta}} &= 0 \\ (f + g_2 - ig_1)P_{\bar{\alpha}} = 0, \quad (f - g_2 - ig_1)P_\alpha = 0, \\ (f - g_2 - ig_1)G_{\bar{\alpha}\beta\gamma} = 0, \quad (f + g_2 - ig_1)G_{\alpha\bar{\beta}\bar{\gamma}} &= 0 \\ G_{-\bar{\beta}_1 \bar{\beta}_2}(f + g_2 - ig_1) - \frac{1}{2}(f - g_2 - ig_1)G_{-\gamma_1 \gamma_2} \epsilon^{\gamma_1 \gamma_2}{}_{\bar{\beta}_1 \bar{\beta}_2} &= 0. \end{aligned} \quad (6.6)$$

The first equation implies that the complex function w is constant. Therefore ϵ_2 is linearly dependent to ϵ_1 over the complex numbers as expected. The last three equations require some explanation. If the Killing spinor ϵ_1 is not pure, then $P_\alpha = P_{\bar{\alpha}} = 0$ and similarly $G_{\bar{\alpha}\beta\gamma} = G_{\alpha\bar{\beta}\bar{\gamma}} = 0$. However if the Killing spinor ϵ_1 is pure, then either $P_{\bar{\alpha}} = 0$ or $P_\alpha = 0$ and similarly either the (1,2) or the (2,1) component of G vanishes depending on whether ϵ_1 is proportional to either 1 or e_{1234} , respectively. In addition, if ϵ_1 is a pure spinor, the last equation implies that either $G_{-\alpha\beta}$ or $G_{-\bar{\alpha}\bar{\beta}}$ will vanish. The Killing spinor equations do not determine the traceless $G_{-\alpha\bar{\beta}}$ component of G .

Next, we summarize the conditions on the flux F . In addition to the self-duality condition on F , we find that

$$iF_{-\alpha\bar{\beta}\gamma}{}^\gamma = \frac{f^2 + g_2^2 + g_1^2}{2fg_2} \Omega_{-,+\bar{\beta}}$$

$$\begin{aligned}
iF_{-+\alpha\beta\gamma} &= -\frac{f^2 - (g_2 - ig_1)^2}{4fg_2}\Omega_{-,+\bar{\delta}}\epsilon^{\bar{\delta}}_{\alpha\beta\gamma} \\
iF_{-\bar{\alpha}_1\bar{\alpha}_2\gamma}{}^\gamma &= -\frac{1}{2fg_2}[-(f^2 + g_2^2 + g_1^2)\Omega_{-, \bar{\alpha}_1\bar{\alpha}_2} \\
&\quad + \frac{1}{2}(f^2 - (g_2 + ig_1)^2)\Omega_{-, \beta_1\beta_2}\epsilon^{\beta_1\beta_2}_{\bar{\alpha}_1\bar{\alpha}_2}] \\
iF_{-\beta_1\beta_2\beta_3\beta_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4} &= -\frac{3}{fg_2}[(\partial_- + \Omega_{-, \gamma}{}^\gamma + \Omega_{-, -+})(f^2 - (g_2 - ig_1)^2)] \\
iF_{-\beta}{}^\beta{}_\gamma{}^\gamma &= \frac{1}{fg_2}[2iQ_-fg_2 - 2ig_2\partial_-g_1 + 2ig_1\partial_-g_2 + \Omega_{-, \gamma}{}^\gamma(f^2 + g_2^2 + g_1^2)] \\
F_{+\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} &= 0, \\
iF_{-+\alpha\bar{\beta}_1\bar{\beta}_2} &= \frac{fg_2}{f^2 + g_2^2 + g_1^2}[\Omega_{\alpha, \bar{\beta}_1\bar{\beta}_2} + 2\Omega_{\gamma, \gamma}{}^\gamma[\bar{\beta}_1g_{\bar{\beta}_2}\alpha]]
\end{aligned} \tag{6.7}$$

and if ϵ_1 is not pure

$$F_{+\alpha\bar{\beta}\gamma}{}^\gamma = 0. \tag{6.8}$$

Alternatively, if ϵ_1 is pure the last equation becomes

$$iF_{+\alpha\bar{\beta}\gamma}{}^\gamma = \pm\Omega_{\alpha, +\bar{\beta}} \tag{6.9}$$

where the sign depends on whether the pure spinor is proportional to 1 or to e_{1234} , respectively. Taking the complex conjugate of the above relation, we find

$$\Omega_{\alpha, +\bar{\beta}} + \Omega_{\bar{\beta}, +\alpha} = 0. \tag{6.10}$$

Observe that in the pure spinor case some of the components of F in (6.7) vanish.

Finally, the conditions on the geometry are

$$\begin{aligned}
\Omega_{+, \alpha\beta} &= 0, \quad iQ_+g_2 + \Omega_{+, \gamma}{}^\gamma f = 0, \\
2\partial_+f + Q_+g_1 + \Omega_{+, -+}f &= 0, \\
2\partial_+g_2 - ig_1\Omega_{+, \gamma}{}^\gamma + \Omega_{+, -+}g_2 &= 0, \\
2\partial_+g_1 - Q_+f + i\Omega_{+, \gamma}{}^\gamma g_2 + g_1\Omega_{+, -+} &= 0, \\
\partial_-(f^2 + g_2^2 + g_1^2) + \Omega_{-, -+}(f^2 + g_2^2 + g_1^2) &= 0, \\
\Omega_{\alpha, +}{}^\alpha &= 0, \quad \Omega_{\alpha, +\bar{\beta}} = 0, \\
4f^2g_2^2\Omega_{\gamma, \gamma}{}^\gamma{}_{\bar{\beta}} + (f^2 + g_2^2 + g_1^2)^2\Omega_{-, +\bar{\beta}} &= 0, \\
\frac{1}{2}(f^2 + g_2^2 + g_1^2)\Omega_{\alpha, \gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} - \Omega_{\alpha, \bar{\beta}_1\bar{\beta}_2}[f^2 - (g_2 - ig_1)^2] &= 0,
\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
Q_\alpha &= \frac{if}{g_2}\Omega_{\alpha, \beta}{}^\beta - i\frac{(g_2 - ig_1)}{fg_2^2}(f^2 + g_1^2 - ig_1g_2)\Omega_{-, +\alpha} \\
\partial_\alpha f &= -i\frac{g_1f}{2g_2}\Omega_{\alpha, \beta}{}^\beta - \frac{1}{2}f\Omega_{\alpha, -+} \\
&\quad + \frac{i(g_2 - ig_1)}{2fg_2^2}(g_1(f^2 + g_1^2 + g_2^2) - ig_2(g_1^2 + g_2^2))\Omega_{-, +\alpha} \\
\partial_\alpha g_1 &= i\frac{(f^2 - g_2^2)}{2g_2}\Omega_{\alpha, \beta}{}^\beta - \frac{1}{2}g_1\Omega_{\alpha, -+}
\end{aligned}$$

$$\begin{aligned}
& - \frac{i}{2g_2^2}(g_2(f^2 - g_2^2) - ig_1(f^2 + g_1^2 + 2g_2^2))\Omega_{-,+\alpha} \\
\partial_\alpha g_2 &= \frac{i}{2}g_1\Omega_{\alpha,\beta}{}^\beta - \frac{1}{2}g_2\Omega_{\alpha,-+} + \frac{1}{2g_2}(f^2 + g_1^2 - ig_1g_2)\Omega_{-,+\alpha}
\end{aligned} \tag{6.12}$$

and if ϵ_1 is not pure

$$\Omega_{\alpha,+\bar{\beta}} = 0 . \tag{6.13}$$

Observe that some of the conditions on the fluxes F can be interpreted as conditions on the geometry because they restrict the ∂_- derivative of functions f, g_1, g_2 which determine the spinor. In turn, the integrability conditions restrict both the fluxes and the geometry.

6.3 The geometry

The linearly independent forms on the spacetime associated with the Killing spinor bi-linears are those that we have computed in [12] for the case of $N = 1$ $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. Because of this, we shall not present them here. It turns out that the vector field X associated to the form

$$\kappa = (f^2 + g_1^2 + g_2^2)e^- \tag{6.14}$$

is a *null Killing* vector field. Moreover one can choose the gauge $f^2 + g_1^2 + g_2^2 = 1$. This is achieved by an $Spin(9,1)$ transformation $e^{a\Gamma_{05}}$ for an appropriate choice of the parameter a . The metric then can be put into the form (5.14). Observe that the first order system for the functions f, g_1, g_2 is again non-linear.

7 Examples of integrability conditions

In this section we will solve the linear systems associated with the integrability conditions for maximal¹⁴ $Spin(7) \ltimes \mathbb{R}^8$ - and $SU(4) \ltimes \mathbb{R}^8$ -backgrounds. This will determine which field equations are implied by the Killing spinor equations. The remaining field equations, which still need to be imposed on the supersymmetric background, will be given explicitly.

There are linear systems associated to the Killing spinor equations and to integrability conditions of any supersymmetric background. However, as it was explained in section 3.2, these systems factorize for maximal H -backgrounds. In particular, the linear system of integrability conditions splits up in three separate parts involving only two types of field equations: E and LF , LP and BP and LG and BG . This considerably simplifies the analysis of the linear systems. In what follows, we shall apply the formalism to the linear systems of the maximal $Spin(7) \ltimes \mathbb{R}^8$ - and $SU(4) \ltimes \mathbb{R}^8$ -backgrounds.

¹⁴The Killing spinor equations for these cases were solved in [13]. The same holds for the case with maximal G_2 supersymmetry, but since this yielded a purely gravitational solution we will not discuss it.

7.1 Maximal $Spin(7) \ltimes \mathbb{R}^8$ -backgrounds

The field equations that are *not* implied by the Killing spinor equations of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$ -backgrounds are (where the tilde indicates traceless components)

$$\begin{aligned}
& E_{--}, LP, BP_{\alpha_1\alpha_2}, \tilde{B}P_{\alpha\bar{\beta}}, BP_{\bar{\beta}_1\bar{\beta}_2}, BP_{\alpha-}, BP_{\bar{\beta}-}, BP_{-+}, LG_{\alpha_1\alpha_2}, \tilde{L}G_{\alpha\bar{\beta}}, LG_{\bar{\beta}_1\bar{\beta}_2}, \\
& LG_{\alpha-}, LG_{\bar{\beta}-}, BG_{\alpha_1\cdots\alpha_4}, BG_{\alpha_1\cdots\alpha_3\bar{\beta}}, BG_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, BG_{\alpha\bar{\beta}_1\cdots\bar{\beta}_3}, BG_{\bar{\beta}_1\cdots\bar{\beta}_4}, \\
& BG_{\alpha_1\cdots\alpha_3-}, BG_{\alpha_1\alpha_2\bar{\beta}-}, BG_{\alpha\bar{\beta}_1\bar{\beta}_2-}, BG_{\bar{\beta}_1\cdots\bar{\beta}_3-}, BG_{\alpha_1\alpha_2-+}, \tilde{B}G_{\alpha\bar{\beta}-+}, BG_{\bar{\beta}_1\bar{\beta}_2-+}, \\
& LF_{\alpha_1\cdots\alpha_4}, LF_{\alpha_1\cdots\alpha_3\bar{\beta}}, LF_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, LF_{\alpha_1\cdots\alpha_3-}, \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}-}, \tilde{L}F_{\alpha\bar{\beta}-+}, \quad (7.1)
\end{aligned}$$

subject to the relations

$$\begin{aligned}
& LP + 2BP_{-+} = BP_{\alpha_1\alpha_2} - \frac{1}{2}\epsilon_{\alpha_1\alpha_2}^{\bar{\beta}_1\bar{\beta}_2}BP_{\bar{\beta}_1\bar{\beta}_2} = 0, \\
& LG_{\alpha_1\alpha_2} + 24BG_{\alpha_1\alpha_2-+} = \tilde{L}G_{\alpha\bar{\beta}} + 24\tilde{B}G_{\alpha\bar{\beta}-+} = 0, \\
& LG_{\bar{\beta}_1\bar{\beta}_2} + 24BG_{\bar{\beta}_1\bar{\beta}_2-+} = LG_{\alpha-} - 24BG_{\alpha\gamma}{}^{\gamma}{}_{-} + 8\epsilon_{\alpha}^{\bar{\beta}_1\cdots\bar{\beta}_3}BG_{\bar{\beta}_1\cdots\bar{\beta}_3-} = 0, \\
& LG_{\bar{\beta}-} + 24BG_{\bar{\beta}\gamma}{}^{\gamma}{}_{-} + 8\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}BG_{\gamma_1\cdots\gamma_3-} = 8BG_{\alpha_1\cdots\alpha_4} + \epsilon_{\alpha_1\cdots\alpha_4}BG_{\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2} = 0, \\
& 8BG_{\bar{\beta}_1\cdots\bar{\beta}_4} + \epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}BG_{\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2} = BG_{\alpha\bar{\beta}_1\cdots\bar{\beta}_3} - \frac{3}{2}g_{\alpha[\bar{\beta}_1}BG_{\bar{\beta}_2\bar{\beta}_3]\gamma}{}^{\gamma} = 0, \\
& BG_{\bar{\beta}\gamma_1\cdots\gamma_3} + \frac{3}{2}g_{\bar{\beta}[\gamma_1}BG_{\gamma_2\gamma_3]\gamma_4}{}^{\gamma_4} = BG_{\alpha_1\alpha_2\gamma}{}^{\gamma} + \frac{1}{2}\epsilon_{\alpha_1\alpha_2}^{\bar{\beta}_1\bar{\beta}_2}BG_{\bar{\beta}_1\bar{\beta}_2\gamma}{}^{\gamma} = 0, \\
& 4BG_{\alpha\bar{\beta}\gamma}{}^{\gamma} - g_{\alpha\bar{\beta}}BG_{\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2} = BG_{\alpha_1\alpha_2-+} - \frac{1}{2}\epsilon_{\alpha_1\alpha_2}^{\bar{\beta}_1\bar{\beta}_2}BG_{\bar{\beta}_1\bar{\beta}_2-+} = 0, \\
& LF_{\alpha\bar{\beta}_1\cdots\bar{\beta}_3} - \frac{3}{2}g_{\alpha[\bar{\beta}_1}LF_{\bar{\beta}_2\bar{\beta}_3]\gamma}{}^{\gamma} = LF_{\alpha_1\alpha_2\gamma}{}^{\gamma} + \frac{1}{2}\epsilon_{\alpha_1\alpha_2}^{\bar{\beta}_1\bar{\beta}_2}LF_{\bar{\beta}_1\bar{\beta}_2\gamma}{}^{\gamma} = 0, \\
& 3LF_{\alpha\bar{\beta}\gamma}{}^{\gamma} + 3\tilde{L}F_{\alpha\bar{\beta}-+} + \epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{\alpha\gamma_1\cdots\gamma_3} = 3LF_{\bar{\beta}\gamma}{}^{\gamma}{}_{-} + \epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{\gamma_1\cdots\gamma_3-} = 0. \quad (7.2)
\end{aligned}$$

Using these relations, one can show that all the field equations and Bianchi identities are satisfied provided one imposes the vanishing of the equations

$$\begin{aligned}
& E_{--}, BP_{\alpha_1\alpha_2}, \tilde{B}P_{\alpha\bar{\beta}}, BP_{\alpha-}, BP_{\bar{\beta}-}, BP_{-+}, \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, LF_{\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2}, LF_{\alpha_1\alpha_2\gamma}{}^{\gamma}, \\
& LF_{\alpha_1\alpha_2\bar{\beta}-}, \tilde{B}G_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, BG_{\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2}, BG_{\alpha_1\alpha_2\gamma}{}^{\gamma}, BG_{\alpha_1\alpha_2\alpha_3-}, BG_{\alpha_1\alpha_2\bar{\beta}-}, BG_{\alpha\bar{\beta}_1\bar{\beta}_2-}, \\
& BG_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3-}, BG_{\alpha_1\alpha_2-+}, \tilde{B}G_{\alpha\bar{\beta}-+}. \quad (7.3)
\end{aligned}$$

Since the Bianchi identity and field equation for F are interchangeable because of the self-duality of F , the only field equation that needs to be imposed is the E_{--} component of the field equations.

We now turn to the corresponding supersymmetric background. As it has been shown in [13], the metric of $Spin(7) \ltimes \mathbb{R}^8$ -backgrounds can be written as

$$ds^2 = 2dv(du + \alpha dv + \beta_I dy^I) + \gamma_{IJ} dy^I dy^J, \quad (7.4)$$

with α , β and γ_{IJ} functions of v and y^I only, and $I = (1, \dots, 8)$. This is a pp-wave metric with rotation, see also [24]. A natural frame is given by

$$e^- = dv, \quad e^+ = du + \alpha dv + \beta_I dy^I, \quad e^\alpha = e^\alpha_I dy^I, \quad e^{\bar{\alpha}} = e^{\bar{\alpha}}_I dy^I. \quad (7.5)$$

The components of the spin connection are

$$\begin{aligned}\Omega_{P,Q-} &= e^I{}_{(P}\partial_v e_{Q)I} + \partial_{[P}\beta_{Q]}, & \Omega_{-,-P} &= \partial_P\alpha - \partial_v\beta_P, \\ \Omega_{-,PQ} &= e^I{}_{[P}\partial_v e_{Q]I} - \partial_{[P}\beta_{Q]},\end{aligned}\tag{7.6}$$

and $\Omega_{P,QR}$, where $P = (\alpha, \bar{\alpha})$. In addition, the components of $\Omega_{P,QR}$ and $\Omega_{-,PQ}$ take values in $spin(7)$, i.e.

$$\Omega_{P,\alpha_1\alpha_2} = \frac{1}{2}\epsilon_{\alpha_1\alpha_2}{}^{\bar{\beta}_1\bar{\beta}_2}\Omega_{P,\bar{\beta}_1\bar{\beta}_2}, \quad \Omega_{P,\alpha}{}^\alpha = 0,\tag{7.7}$$

and similarly for $\Omega_{-,QR}$.

The Killing spinor equations restrict the fluxes as follows. The non-vanishing components of the fluxes are P_- , G_{PQ-} and $F_{P_1\dots P_4-}$ which in addition satisfy the following conditions. G_{PQ-} takes values in $spin(7)$, i.e. in the decomposition $\Lambda^2(\mathbb{R}^8) = \Lambda^2_7(\mathbb{R}^8) \oplus \Lambda^2_{21}(\mathbb{R}^8)$ into $Spin(7)$ representations, only the $\Lambda^2_{21}(\mathbb{R}^8)$ is allowed by the Killing spinor equations. Similarly, the components of $F_{P_1\dots P_4-}$ lie in $\Lambda^4_1(\mathbb{R}^8)$ and $\Lambda^4_{27}(\mathbb{R}^8)$ in the decomposition $\Lambda^4(\mathbb{R}^8) = \Lambda^4_1(\mathbb{R}^8) \oplus \Lambda^4_7(\mathbb{R}^8) \oplus \Lambda^4_{27}(\mathbb{R}^8) \oplus \Lambda^4_{35}(\mathbb{R}^8)$. The singlet is given by

$$F_{P_1\dots P_4-}\psi^{P_1\dots P_4} = 24Q_-, \tag{7.8}$$

where ψ is the $Spin(7)$ -invariant four-form, whose definition can be found in [13].

Among the field equations that still need to be imposed on the solution to the $N = 2$ $Spin(7) \ltimes \mathbb{R}^8$ Killing spinor equations is the Einstein equation E_{--} , which is given by

$$\begin{aligned}-(\partial^P + \Omega_Q{}^{QP})(\partial_P\alpha - \partial_v\beta_P) + \partial_{[P}\beta_{Q]}\partial^P\beta^Q - \frac{1}{2}\gamma^{IJ}\partial_v{}^2\gamma_{IJ} - \frac{1}{4}\partial_v\gamma^{IJ}\partial_v\gamma_{IJ} \\ - \frac{1}{6}F_{-P_1\dots P_4}F_-{}^{P_1\dots P_4} - \frac{1}{4}G_-{}^{P_1P_2}G_{-P_1P_2}^* - 2P_-P_-^* = 0.\end{aligned}\tag{7.9}$$

where γ^{IJ} is the inverse of the metric γ_{IJ} defined in (7.4). For the special case of α , β_I and γ_{IJ} independent of v this equation becomes

$$-\square_8\alpha + \partial_{[P}\beta_{Q]}\partial^P\beta^Q - \frac{1}{6}F_{-P_1\dots P_4}F_-{}^{P_1\dots P_4} - \frac{1}{4}G_-{}^{P_1P_2}G_{-P_1P_2}^* - 2P_-P_-^* = 0, \tag{7.10}$$

where \square_8 is the Laplacian on the eight-dimensional space and $\partial_{[P}\beta_{Q]}$ only takes values in $Spin(7)$.

In addition one needs to impose several components of the Bianchi identities on the fluxes P_- , G_{PQ-} and $F_{P_1\dots P_4-}$. For example, the remaining BP components imply that P_- is a function of v only. We will not analyze the rest of these restrictions in detail.

Observe that the contribution of the rotation in the Einstein equations has a different sign from that of the contribution of the fluxes. Because of this and assuming that the transverse space of the pp-wave is a compact $Spin(7)$ manifold, the Einstein equation can be solved provided that the total rotation cancels the contributions from the fluxes. This means that the integral of the expression in the right-hand-side of (7.10) must vanish. For a detailed similar argument see [12]. The above solution resembles flux-tube type of configurations [25] but without the backreaction of the branes. There are also similarities with the solutions of [26, 27].

7.2 Maximal $SU(4) \ltimes \mathbb{R}^8$ backgrounds

The field equations that are *not* implied by the Killing spinor equations of maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ -backgrounds are

$$\begin{aligned} E_{--}, \quad LP, \quad \tilde{B}P_{\alpha\bar{\beta}}, \quad BP_{\alpha-}, \quad BP_{\bar{\beta}-}, \quad BP_{-+}, \quad \tilde{L}G_{\alpha\bar{\beta}}, \quad LG_{\alpha-}, \quad LG_{\bar{\beta}-}, \\ \tilde{B}G_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, \quad BG_{\alpha_1\alpha_2\bar{\beta}-}, \quad BG_{\alpha\bar{\beta}_1\bar{\beta}_2-}, \quad \tilde{B}G_{\alpha\bar{\beta}-+}, \quad \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, \quad \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}-}, \end{aligned} \quad (7.11)$$

subject to the relations

$$\begin{aligned} LP + 2BP_{-+} &= \tilde{L}G_{\alpha\bar{\beta}} + 24\tilde{B}G_{\alpha\bar{\beta}-+} = 0, \\ LG_{\alpha-} - 24BG_{\alpha\gamma}{}^{\gamma}{}_{-} &= LG_{\bar{\beta}-} + 24BG_{\bar{\beta}\gamma}{}^{\gamma}{}_{-} = 0. \end{aligned} \quad (7.12)$$

Using these relations, one can show that all field equations are satisfied provided that the field equations

$$\begin{aligned} E_{--}, \quad \tilde{B}P_{\alpha\bar{\beta}}, \quad BP_{\alpha-}, \quad BP_{\bar{\beta}-}, \quad BP_{-+}, \quad \tilde{B}G_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, \quad BG_{\alpha_1\alpha_2\bar{\beta}-}, \\ BG_{\alpha\bar{\beta}_1\bar{\beta}_2-}, \quad \tilde{B}G_{\alpha\bar{\beta}-+}, \quad \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}, \quad \tilde{L}F_{\alpha_1\alpha_2\bar{\beta}-}, \end{aligned} \quad (7.13)$$

are satisfied. It is easy to see that apart from the Bianchi identities, the only field equation that one has to impose is the E_{--} component of the Einstein equation.

The investigation of the field equations of the maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds is related to that of maximal $Spin(7) \ltimes \mathbb{R}^8$ -backgrounds. In particular, there is a gauge for the Killing spinors such that $\Omega_{A,-+} = 0$ and $\Omega_{A,\beta}{}^{\beta} = 0$ [13]. In addition the metric can be written in Penrose coordinates as in (7.4) and so one can introduce the frame (7.5). One can compute the spin connection which has components $\Omega_{-,-P}$, $\Omega_{P,Q-}$, $\Omega_{-,PQ}$ and $\Omega_{P,QR}$. The first three are given in (7.6), and in addition the latter two take values in $SU(4)$, i.e.

$$\Omega_{P,\alpha_1\alpha_2} = 0, \quad \Omega_{P,\alpha}{}^{\alpha} = 0, \quad (7.14)$$

and similarly for $\Omega_{-,PQ}$. The non-vanishing components of the fluxes are

$$P_{-}, \quad \tilde{G}_{\alpha\bar{\beta}-}, \quad G_{\alpha}{}^{\alpha}{}_{-}(v), \quad F_{\alpha_1\cdots\alpha_4-}(v), \quad \tilde{F}_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2-}, \quad F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}{}_{-} = 2Q_{-}. \quad (7.15)$$

Using these, one can easily compute the Einstein equation E_{--} . It is easy to see that it takes the same form as (7.9). In addition the remaining Bianchi identities will impose closure of the remaining fluxes P_{-} , G_{PQ-} and $F_{P_1\cdots P_4-}$.

In the case that the components of the metric are independent of the v coordinate, the transverse space of the pp-wave is a Calabi-Yau manifold. To find a solution, one can use the Donaldson theorem for $U(1)$ connections and require cancelation of the rotation and flux charges. Using a similar argument to that in [12], one can show that there is a smooth solution for pp-waves with transverse space a compact Calabi-Yau manifold.

8 Concluding remarks

We have shown that the Killing spinor equations of any IIB supergravity background can be written as a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. This has been achieved by using the spinorial geometry techniques of [14]. We have also shown that another linear system, constructed in a similar way, can be used to determine the field equations and Bianchi identities of IIB supergravity that are determined by the Killing spinors for any supersymmetric background. These two linear systems can be used to systematically investigate all supersymmetric backgrounds of IIB supergravity.

For general supersymmetric backgrounds these two linear systems are rather complicated. However, we have shown that these linear systems simplify for backgrounds that admit H -invariant spinors, $H \subset Spin(9, 1)$. We have mostly focused on two cases, those for which the background admits a maximal number of H -invariant Killing spinors, maximally supersymmetric H -backgrounds, and those that admit half the number of maximal H -invariant spinors, half-maximally supersymmetric H -backgrounds. In the former case, the Killing spinor equations factorize and the resulting linear systems are easy to solve. We have demonstrated that the system associated with the Killing spinor equations gives rise to a flatness condition for the connection which is identified as the restriction of the supercovariant connection on the bundle of Killing spinors \mathcal{K} .

There are several cases of half-maximal H -backgrounds which should be considered. The generic case consists of those backgrounds for which the Killing spinors are linearly independent over the complex numbers. There are also several degenerate cases for which the Killing spinors are linearly dependent over the complex numbers but linearly independent over the real numbers. The degenerate cases are of co-dimension two or more relative to the generic case. We have demonstrated that the Killing spinor equations of half-maximal H -backgrounds do not factorize. The linear system of the Killing spinor equations gives rise to a flatness condition for the restriction of the supercovariant connection on the bundle of Killing spinors \mathcal{K} . However, the restricted connection depends non-linearly on the functions that determine the Killing spinors.

To give an overview of the current status of the problem in IIB supergravity, we summarize some of the results in the table 1 below. In this table, we indicate the cases that have been investigated as well as the maximal and half-maximal H -backgrounds that remain to be tackled.

In table 1 below, the list of cases that remain to be tackled contains the $N = 4$ and $N = 8$ $SU(3)$ -backgrounds. It is expected that the former includes all backgrounds which are dual to $\mathcal{N} = 1$ ($N = 4$) four-dimensional gauge theories. The list also includes all supersymmetric backgrounds that preserve $1/2$ of the supersymmetry ($N = 16$). There are three classes of $1/2$ supersymmetric backgrounds. The maximal \mathbb{R}^8 -backgrounds, the maximal $SU(2)$ -backgrounds and the half-maximal 1-backgrounds. It would be of interest to investigate all these cases.

H	N = 1	N = 2	N = 3	N = 4	N = 6	N = 8	N = 16	N = 32
$Spin(7) \ltimes \mathbb{R}^8$	✓	✓	-	-	-	-	-	-
$SU(4) \ltimes \mathbb{R}^8$	✓	✓	-	✓	-	-	-	-
G_2	✓	⊙	-	✓	-	-	-	-
$Sp(2) \ltimes \mathbb{R}^8$	-	-	⊙	-	⊙	-	-	-
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	-	-	-	⊙	-	⊙	-	-
$SU(3)$	-	-	-	⊙	-	⊙	-	-
\mathbb{R}^8	-	-	-	-	-	⊙	⊙	-
$SU(2)$	-	-	-	-	-	⊙	⊙	-
1	-	-	-	-	-	-	⊙	✓

Table 1: ✓ denotes the cases for which the Killing spinor equations have already been solved. ⊙ denotes the remaining cases that correspond to backgrounds with H -invariant spinors and can be tackled with the techniques described in this paper. – denotes the cases that do not occur, e.g. there are no backgrounds with $N > 2$ and $Spin(7) \ltimes \mathbb{R}^8$ -invariant Killing spinors. The remaining entries may occur but it is expected that the associated linear systems are more involved.

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Appendix A Spinors

The description of IIB supergravity spinors that we used in this paper can be found in [12]. For the general theory see [28, 29, 30]. Here for completeness, we shall briefly summarize the main formulae without explanation.

Consider the vector space $U = \mathbb{R} \langle e_1, \dots, e_5 \rangle$, where e_1, \dots, e_5 is an orthonormal basis. The space of Dirac spinors of $Spin(9, 1)$ is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$. This decomposes into two complex chiral representations according to the degree of the form $\Delta_c^+ = \Lambda^{\text{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\text{odd}}(U \otimes \mathbb{C})$. These are the complex Weyl representations of $Spin(9, 1)$. The gamma matrices are represented on Δ_c as

$$\begin{aligned}
\Gamma_0 \eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta, & \Gamma_5 \eta &= e_5 \wedge \eta + e_5 \lrcorner \eta, \\
\Gamma_i \eta &= e_i \wedge \eta + e_i \lrcorner \eta, & i &= 1, \dots, 4 \\
\Gamma_{5+i} \eta &= ie_i \wedge \eta - ie_i \lrcorner \eta,
\end{aligned} \tag{A.1}$$

where Γ_0 is the gamma matrix along the time direction. The above gamma matrices satisfy the Clifford algebra relations $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}$ with respect to the Lorentzian inner product as expected.

The Dirac inner product on the space of spinors Δ_c is

$$D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle, \tag{A.2}$$

where

$$< z^a e_a, w^b e_b > = \sum_{a=1}^5 (z^a)^* w^a , \quad (\text{A.3})$$

on $U \otimes \mathbb{C}$ and then extended to Δ_c . Note that $(z^a)^*$ is the standard complex conjugate of z^a .

The Majorana $Spin(9, 1)$ -invariant inner product that we use is

$$B(\eta, \theta) = < B(\eta^*), \theta > , \quad (\text{A.4})$$

where $B = \Gamma_{06789}$. The Majorana-Weyl representations Δ^\pm of $Spin(9, 1)$ are constructed by imposing the reality condition

$$\eta = -\Gamma_0 B(\eta^*) , \quad (\text{A.5})$$

or equivalently

$$\eta^* = \Gamma_{6789} \eta , \quad (\text{A.6})$$

on Δ_c^\pm . The map $C = \Gamma_{6789}$ is the charge conjugation matrix.

The oscillator basis in Δ_c that we use in this paper is

$$\Gamma_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\Gamma_{\alpha} + i\Gamma_{\alpha+5}) , \quad \Gamma_{\pm} = \frac{1}{\sqrt{2}}(\Gamma_5 \pm \Gamma_0) , \quad \Gamma_{\alpha} = \frac{1}{\sqrt{2}}(\Gamma_{\alpha} - i\Gamma_{\alpha+5}) . \quad (\text{A.7})$$

Observe that the Clifford algebra relations in the above basis are $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}$, where the non-vanishing components of the metric are $\eta_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$, $\eta_{+-} = 1$. In addition, we define $\Gamma^B = g^{BA}\Gamma_A$. The 1 spinor is a Clifford vacuum, $\Gamma^{\alpha}1 = \Gamma^{-1}1 = 0$ and the representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^{\bar{\alpha}}, \Gamma^+$ or equivalently any spinor can be written as

$$\eta = \sum_{k=0}^5 \frac{1}{k!} \phi_{\bar{a}_1 \dots \bar{a}_k} \Gamma^{\bar{a}_1 \dots \bar{a}_k} 1 , \quad \bar{a} = \bar{\alpha}, + , \quad (\text{A.8})$$

i.e. $\Gamma^{\bar{a}_1 \dots \bar{a}_k} 1$, for $k = 0, \dots, 5$, is a basis in the space of (Dirac) spinors.

The spacetime form bi-linears associated with the pair of spinors (η, ϵ) . are

$$\alpha(\eta, \epsilon) = \frac{1}{k!} B(\eta, \Gamma_{A_1 \dots A_k} \epsilon) e^{A_1} \wedge \dots \wedge e^{A_k} , \quad k = 0, \dots, 9 . \quad (\text{A.9})$$

For the application to backgrounds with $SU(4) \ltimes \mathbb{R}^8$ spinors, we use the form bi-linear of 1 and e_{1234} spinors. These are the following (see also [12]): a one-form

$$\kappa(e_{1234}, 1) = \kappa(1, e_{1234}) = e^0 - e^5 , \quad (\text{A.10})$$

a three-form

$$\xi(e_{1234}, 1) = -\xi(1, e_{1234}) = i(e^0 - e^5) \wedge \omega , \quad (\text{A.11})$$

and five-forms

$$\tau(1, 1) = (e^0 - e^5) \wedge \chi$$

$$\begin{aligned}\tau(e_{1234}, e_{1234}) &= (e^0 - e^5) \wedge \chi^* \\ \tau(e_{1234}, 1) &= \tau(1, e_{1234}) = -\frac{1}{2}(e^0 - e^5)\omega \wedge \omega ,\end{aligned}\tag{A.12}$$

where

$$\begin{aligned}\omega &= e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9 \\ \chi &= (e^1 + ie^6) \wedge (e^2 + ie^7) \wedge (e^3 + ie^8) \wedge (e^4 + ie^9) .\end{aligned}\tag{A.13}$$

Note that χ and ω are the familiar $SU(4)$ invariant forms.

Using the above results, we can easily compute the spacetime form bilinears of the half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ backgrounds. The Killing spinors are $\epsilon = z\eta$, where $\eta_1 = 1 + e_{1234}$ and $\eta_2 = i(1 - e_{1234})$. Since the Killing spinors ϵ_1 and ϵ_2 are generically complex, $\tilde{\epsilon}_1 = C(\epsilon_1)^*$ and $\tilde{\epsilon}_2 = C(\epsilon_2)^*$ are also defined on the spacetime although they are not necessarily Killing. Because of this, the forms that are defined on the spacetime are associated with the bi-linears (ϵ_i, ϵ_j) , $(\epsilon_i, \tilde{\epsilon}_j)$ and $(\tilde{\epsilon}_i, \tilde{\epsilon}_j)$, $i, j = 1, 2$. It suffices to compute the forms associated with the first two bilinears because the forms of the last bilinear can be easily computed from those of the first. To see this observe that since $\epsilon = z\eta$ and η_i are Majorana-Weyl spinors, then $\tilde{\epsilon} = z^*\eta$. Thus the effect of the charge conjugation operation is to replace the matrix z with its complex conjugate z^* . In particular, we find the one-forms

$$\begin{aligned}\kappa(\epsilon_1, \epsilon_1) &= 2(z_{11}^2 + z_{12}^2)(e^0 - e^5) , \\ \kappa(\epsilon_2, \epsilon_2) &= 2(z_{21}^2 + z_{22}^2)(e^0 - e^5) , \\ \kappa(\epsilon_1, \epsilon_2) &= 2(z_{11}z_{21} + z_{12}z_{22})(e^0 - e^5) , \\ \kappa(\epsilon_1, \tilde{\epsilon}_1) &= 2(|z_{11}|^2 + |z_{12}|^2)(e^0 - e^5) , \\ \kappa(\epsilon_2, \tilde{\epsilon}_2) &= 2(|z_{21}|^2 + |z_{22}|^2)(e^0 - e^5) , \\ \kappa(\epsilon_1, \tilde{\epsilon}_2) &= 2(z_{11}z_{21}^* + z_{12}z_{22}^*)(e^0 - e^5) , \\ \kappa(\tilde{\epsilon}_1, \epsilon_2) &= 2(z_{11}^*z_{21} + z_{12}^*z_{22})(e^0 - e^5) ,\end{aligned}\tag{A.14}$$

the three-forms

$$\begin{aligned}\xi(\epsilon_1, \epsilon_2) &= -2\det z(e^0 - e^5) \wedge \omega , \\ \xi(\epsilon_1, \tilde{\epsilon}_1) &= 4i\text{Im}(z_{11}^*z_{12})(e^0 - e^5) \wedge \omega , \\ \xi(\epsilon_1, \tilde{\epsilon}_2) &= -2(z_{11}z_{22}^* - z_{12}z_{21}^*)(e^0 - e^5) \wedge \omega , \\ \xi(\tilde{\epsilon}_1, \epsilon_2) &= -2(z_{11}^*z_{22} - z_{12}^*z_{21})(e^0 - e^5) \wedge \omega ,\end{aligned}\tag{A.15}$$

and the five-forms

$$\begin{aligned}\tau(\epsilon_1, \epsilon_1) &= (e^0 - e^5) \wedge [(z_{11} + iz_{12})^2\chi + (z_{11} - iz_{12})^2\chi^* - (z_{11}^2 + z_{12}^2)\omega \wedge \omega] , \\ \tau(\epsilon_2, \epsilon_2) &= (e^0 - e^5) \wedge [(z_{21} + iz_{22})^2\chi + (z_{21} - iz_{22})^2\chi^* - (z_{21}^2 + z_{22}^2)\omega \wedge \omega] , \\ \tau(\epsilon_1, \epsilon_2) &= (e^0 - e^5) \wedge [(z_{11} + iz_{12})(z_{21} + iz_{22})\chi + (z_{11} - iz_{12})(z_{21} - iz_{22})\chi^* \\ &\quad - (z_{11}z_{21} + z_{12}z_{22})\omega \wedge \omega] , \\ \tau(\epsilon_1, \tilde{\epsilon}_1) &= (e^0 - e^5) \wedge [(z_{11} + iz_{12})(z_{11}^* + iz_{12}^*)\chi + (z_{11} - iz_{12})(z_{11}^* - iz_{12}^*)\chi^* \\ &\quad - (|z_{11}|^2 + |z_{12}|^2)\omega \wedge \omega] , \\ \tau(\epsilon_2, \tilde{\epsilon}_2) &= (e^0 - e^5) \wedge [(z_{21} + iz_{22})(z_{21}^* + iz_{22}^*)\chi + (z_{21} - iz_{22})(z_{21}^* - iz_{22}^*)\chi^* \\ &\quad - (|z_{21}|^2 + |z_{22}|^2)\omega \wedge \omega] ,\end{aligned}$$

$$\begin{aligned}
\tau(\epsilon_1, \tilde{\epsilon}_2) &= (e^0 - e^5) \wedge [(z_{11} + iz_{12})(z_{21}^* + iz_{22}^*)\chi + (z_{11} - iz_{12})(z_{21}^* - iz_{22}^*)\chi^* \\
&\quad - (z_{11}z_{21}^* + z_{12}z_{22}^*)\omega \wedge \omega] , \\
\tau(\tilde{\epsilon}_1, \epsilon_2) &= (e^0 - e^5) \wedge [(z_{11}^* + iz_{12}^*)(z_{21} + iz_{22})\chi + (z_{11}^* - iz_{12}^*)(z_{21} - iz_{22})\chi^* \\
&\quad - (z_{11}^*z_{21} + z_{12}^*z_{22})\omega \wedge \omega] .
\end{aligned} \tag{A.16}$$

It is worth mentioning that all the one-forms are along the same null direction. The same applies for the three-forms. However, the five-forms point to different directions spanned by $(e^0 - e^5) \wedge \chi$, $(e^0 - e^5) \wedge \chi^*$ and $(e^0 - e^5) \wedge \omega \wedge \omega$.

Appendix B Killing spinor equations

B.1 Killing spinor equations on 1

The first spinor basis element we consider is 1. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.8), we find

$$\begin{aligned}
1 &: 0 , \\
\Gamma^{\bar{\beta}} &: \frac{1}{4}G_{\bar{\beta}\gamma}{}^{\gamma} + \frac{1}{4}G_{\bar{\beta}-+} , \\
\Gamma^+ &: \frac{1}{4}G_{+\gamma}{}^{\gamma} , \\
\Gamma^{(2)} &: 0 , \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3} &: \frac{1}{24}G_{\bar{\beta}_1 \dots \bar{\beta}_3} - \frac{1}{12}\epsilon_{\bar{\beta}_1 \dots \bar{\beta}_3}{}^{\gamma} P_{\gamma} , \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2 +} &: \frac{1}{8}G_{\bar{\beta}_1 \bar{\beta}_2 +} , \\
\Gamma^{(4)} &: 0 , \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4 +} &: \frac{1}{96}\epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4} P_{+} .
\end{aligned} \tag{B.1}$$

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the α -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: D_{\alpha} + \frac{1}{2}\Omega_{\alpha,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{\alpha,-+} + \frac{i}{4}F_{\alpha\gamma_1}{}^{\gamma_1\gamma_2}{}^{\gamma_2} + \frac{i}{2}F_{\alpha\gamma_1}{}^{\gamma_1}{}_{-+} , \\
\Gamma^{(1)} &: 0 , \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2} &: \frac{1}{4}\Omega_{\alpha,\bar{\beta}_1 \bar{\beta}_2} + \frac{i}{4}F_{\alpha\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma} + \frac{i}{4}F_{\alpha\bar{\beta}_1 \bar{\beta}_2 -+} - \frac{1}{32}\epsilon_{\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma_1 \gamma_2} G_{\alpha\gamma_1 \gamma_2} , \\
\Gamma^{\bar{\beta}+} &: \frac{1}{2}\Omega_{\alpha,\bar{\beta}+} + \frac{i}{2}F_{\alpha\bar{\beta}+}{}^{\gamma} , \\
\Gamma^{(3)} &: 0 , \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4} &: \frac{i}{48}F_{\alpha\bar{\beta}_1 \dots \bar{\beta}_4} - \frac{1}{768}\epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4}(G_{\alpha\gamma}{}^{\gamma} - G_{\alpha-+}) , \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3 +} &: \frac{i}{12}F_{\alpha\bar{\beta}_1 \dots \bar{\beta}_3 +} - \frac{1}{96}\epsilon_{\bar{\beta}_1 \dots \bar{\beta}_3}{}^{\gamma} G_{\alpha\gamma +} , \\
\Gamma^{(5)} &: 0 .
\end{aligned} \tag{B.2}$$

Along the $\bar{\alpha}$ -frame derivative of the supercovariant connection we find

$$\begin{aligned}
1 &: D_{\bar{\alpha}} + \frac{1}{2}\Omega_{\bar{\alpha},\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{\bar{\alpha},-+} + \frac{i}{4}F_{\bar{\alpha}\gamma_1}{}^{\gamma_1\gamma_2}{}^{\gamma_2} + \frac{i}{2}F_{\bar{\alpha}\gamma_1}{}^{\gamma_1}{}_{-+} - \frac{1}{24}\epsilon_{\bar{\alpha}}{}^{\gamma_1 \dots \gamma_3} G_{\gamma_1 \dots \gamma_3} , \\
\Gamma^{(1)} &: 0 , \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2} &: \frac{1}{4}\Omega_{\bar{\alpha},\bar{\beta}_1 \bar{\beta}_2} + \frac{i}{4}F_{\bar{\alpha}\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma} + \frac{i}{4}F_{\bar{\alpha}\bar{\beta}_1 \bar{\beta}_2 -+} - \frac{1}{32}\epsilon_{\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma_1 \gamma_2} (2G_{\bar{\alpha}\gamma_1 \gamma_2} + g_{\bar{\alpha}\gamma_1} G_{\gamma_2 \delta}{}^{\delta} - g_{\bar{\alpha}\gamma_1} G_{\gamma_2 -+}) , \\
\Gamma^{\bar{\beta}+} &: \frac{1}{2}\Omega_{\bar{\alpha},\bar{\beta}+} + \frac{i}{2}F_{\bar{\alpha}\bar{\beta}+}{}^{\gamma} + \frac{1}{16}\epsilon_{\bar{\alpha}\bar{\beta}}{}^{\gamma_1 \gamma_2} G_{\gamma_1 \gamma_2 +} ,
\end{aligned}$$

$$\begin{aligned}
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4} &: -\frac{1}{384} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4} (G_{\bar{\alpha} \gamma}{}^\gamma - G_{\bar{\alpha} - +}), \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3 +} &: \frac{i}{12} F_{\bar{\alpha} \bar{\beta}_1 \dots \bar{\beta}_3 +} - \frac{1}{192} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_3}{}^\gamma (4G_{\bar{\alpha} \gamma +} + g_{\bar{\alpha} \gamma} G_{+ \delta}{}^\delta), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.3}$$

The supercovariant derivative with $M = -$ gives

$$\begin{aligned}
1 &: D_- + \frac{1}{2} \Omega_{-, \gamma}{}^\gamma + \frac{1}{2} \Omega_{-, -+} + \frac{i}{4} F_{-\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2} &: \frac{1}{4} \Omega_{-, \bar{\beta}_1 \bar{\beta}_2} + \frac{i}{4} F_{-\bar{\beta}_1 \bar{\beta}_2 \gamma}{}^\gamma - \frac{1}{16} \epsilon_{\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma_1 \gamma_2} G_{-\gamma_1 \gamma_2}, \\
\Gamma^{\bar{\beta}+} &: \frac{1}{2} \Omega_{-, \bar{\beta}+} - \frac{i}{2} F_{\bar{\beta}-+ \gamma}{}^\gamma + \frac{1}{48} \epsilon_{\bar{\beta}}{}^{\gamma_1 \dots \gamma_3} G_{\gamma_1 \dots \gamma_3}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4} &: \frac{i}{48} F_{-\bar{\beta}_1 \dots \bar{\beta}_4} - \frac{1}{384} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4} G_{-\gamma}{}^\gamma, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3 +} &: -\frac{i}{12} F_{\bar{\beta}_1 \dots \bar{\beta}_3 -+} + \frac{1}{192} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_3}{}^\gamma (G_{\gamma \delta}{}^\delta + 3G_{\gamma - +}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.4}$$

Finally, for $M = +$ we find

$$\begin{aligned}
1 &: D_+ + \frac{1}{2} \Omega_{+, \gamma}{}^\gamma + \frac{1}{2} \Omega_{+, -+} + \frac{i}{4} F_{+\gamma_1}{}^{\gamma_1}{}_{\gamma_2}{}^{\gamma_2}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2} &: \frac{1}{4} \Omega_{+, \bar{\beta}_1 \bar{\beta}_2} + \frac{i}{4} F_{+\bar{\beta}_1 \bar{\beta}_2 \gamma}{}^\gamma - \frac{1}{32} \epsilon_{\bar{\beta}_1 \bar{\beta}_2}{}^{\gamma_1 \gamma_2} G_{+\gamma_1 \gamma_2}, \\
\Gamma^{\bar{\beta}+} &: \frac{1}{2} \Omega_{+, \bar{\beta}+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4} &: \frac{i}{48} F_{+\bar{\beta}_1 \dots \bar{\beta}_4} - \frac{1}{768} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4} G_{+\gamma}{}^\gamma, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3 +} &: 0, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.5}$$

B.2 Killing spinor equations on e_{ij}

Next we consider the basis elements e_{ij} with $i, j \leq 4$, i.e. without holomorphic indices. We split up α into $a = (i, j)$ and p , which contains the remaining two holomorphic indices. Furthermore, two different two-dimensional Levi-Civita tensors will appear, which are defined by $\epsilon_{ij} = +1$ and $\epsilon_{p_1 p_2} = \epsilon_{ij p_1 p_2}$. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.8), we find

$$\begin{aligned}
1 &: 0, \\
\Gamma^{\bar{b}} &: \frac{1}{4} \epsilon_{\bar{b}}{}^{c_1} (G_{c_1 c_2}{}^{c_2} - G_{c_1 r}{}^r - G_{c_1 - +}), \\
\Gamma^{\bar{q}} &: \epsilon_{\bar{q}}{}^r P_r - \frac{1}{4} \epsilon^{c_1 c_2} G_{c_1 c_2 \bar{q}}, \\
\Gamma^+ &: -\frac{1}{4} \epsilon^{c_1 c_2} G_{c_1 c_2 +}, \\
\Gamma^{(2)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: -\frac{1}{16} \epsilon_{\bar{b}_1 \bar{b}_2} (G_{\bar{q} c}{}^c - G_{\bar{q} r}{}^r - G_{\bar{q} - +}), \\
\Gamma^{\bar{b}_1 \bar{b}_2 +} &: -\frac{1}{16} (G_{+ c}{}^c - G_{+ r}{}^r), \\
\Gamma^{\bar{b} \bar{q}_1 \bar{q}_2} &: -\frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} P_{\bar{b}} - \frac{1}{8} \epsilon_{\bar{b}}{}^c G_{c \bar{q}_1 \bar{q}_2},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\bar{b}\bar{q}+} &: -\frac{1}{4}\epsilon_{\bar{b}}^c G_{c\bar{q}+}, \\
\Gamma^{\bar{q}_1\bar{q}_2+} &: -\frac{1}{4}\epsilon_{\bar{q}_1\bar{q}_2} P_+, \\
\Gamma^{(4)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}_1\bar{q}_2+} &: \frac{1}{32}\epsilon_{\bar{b}_1\bar{b}_2} G_{\bar{q}_1\bar{q}_2+}.
\end{aligned} \tag{B.6}$$

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the a -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: -\frac{1}{2}\epsilon^{c_1c_2}\Omega_{a,c_1c_2} + \frac{1}{4}\epsilon^{r_1r_2}G_{ar_1r_2}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2} &: \frac{1}{4}\epsilon_{\bar{b}_1\bar{b}_2}(D_a - \frac{1}{2}\Omega_{a,c}^c + \frac{1}{2}\Omega_{a,r}^r + \frac{1}{2}\Omega_{a,-+} - \frac{i}{2}F_{ac}^c{}^r - \frac{i}{2}F_{a-+}^c + \frac{i}{4}F_{ar_1}{}^{r_1}{}_{r_2}{}^{r_2} \\
&\quad + \frac{i}{2}F_{a-+}{}^r) - \frac{1}{16}\epsilon^{r_1r_2}g_{a[\bar{b}_1}G_{\bar{b}_2]r_1r_2}, \\
\Gamma^{\bar{b}\bar{q}} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(\Omega_{a,c\bar{q}} + iF_{ac\bar{q}}{}^r + iF_{ac\bar{q}-+}) + \frac{1}{16}g_{a\bar{b}}\epsilon_{\bar{q}}^{r_1}(3G_{r_1c}^c + 3G_{r_1r_2}{}^{r_2} - G_{r_1-+}) + \frac{1}{2}\epsilon_{\bar{q}}^r G_{a\bar{b}r}, \\
\Gamma^{\bar{b}+} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(\Omega_{a,c+} + iF_{ac+}{}^r) - \frac{1}{16}g_{a\bar{b}}\epsilon^{r_1r_2}G_{r_1r_2+}, \\
\Gamma^{\bar{q}_1\bar{q}_2} &: -\frac{1}{16}\epsilon_{\bar{q}_1\bar{q}_2}(G_{ac}^c - G_{ar}^r + G_{a-+}), \\
\Gamma^{\bar{q}+} &: \frac{1}{4}\epsilon_{\bar{q}}^r G_{ar+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}_1\bar{q}_2} &: \frac{1}{16}\epsilon_{\bar{b}_1\bar{b}_2}(\Omega_{a,\bar{q}_1\bar{q}_2} - iF_{a\bar{q}_1\bar{q}_2}{}^r + iF_{a\bar{q}_1\bar{q}_2-+}) - \frac{1}{64}\epsilon_{\bar{q}_1\bar{q}_2}g_{a[\bar{b}_1}(3G_{\bar{b}_2]c}^c + G_{\bar{b}_2]r}^r - G_{\bar{b}_2]-+}), \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}+} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2}(\Omega_{a,\bar{q}+} - iF_{a\bar{q}+}{}^c + iF_{a\bar{q}+}{}^r) - \frac{1}{16}\epsilon_{\bar{q}}^r g_{a[\bar{b}_1}G_{\bar{b}_2]r+}, \\
\Gamma^{\bar{b}\bar{q}_1\bar{q}_2+} &: -\frac{i}{4}\epsilon_{\bar{b}}^c F_{ac\bar{q}_1\bar{q}_2+} - \frac{1}{64}\epsilon_{\bar{q}_1\bar{q}_2}(4G_{a\bar{b}+} - g_{a\bar{b}}(G_{+c}^c - G_{+r}^r)), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.7}$$

Along the \bar{a} -frame derivative of the supercovariant connection we find

$$\begin{aligned}
1 &: -\frac{1}{2}\epsilon^{c_1c_2}(\Omega_{\bar{a},c_1c_2} + iF_{\bar{a}c_1c_2}{}^r + iF_{\bar{a}c_1c_2-+}) + \frac{1}{8}\epsilon^{r_1r_2}G_{\bar{a}r_1r_2}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2} &: \frac{1}{4}\epsilon_{\bar{b}_1\bar{b}_2}(D_{\bar{a}} - \frac{1}{2}\Omega_{\bar{a},c}^c + \frac{1}{2}\Omega_{\bar{a},r}^r + \frac{1}{2}\Omega_{\bar{a},-+} - \frac{i}{2}F_{\bar{a}c}^c{}^r - \frac{i}{2}F_{\bar{a}-+}^c + \frac{i}{4}F_{\bar{a}r_1}{}^{r_1}{}_{r_2}{}^{r_2} + \frac{i}{2}F_{\bar{a}-+}{}^r), \\
\Gamma^{\bar{b}\bar{q}} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(\Omega_{\bar{a},c\bar{q}} - iF_{\bar{a}c\bar{q}}^c + iF_{\bar{a}c\bar{q}}{}^r + iF_{\bar{a}c\bar{q}-+}) + \frac{1}{8}\epsilon_{\bar{q}}^r G_{\bar{a}\bar{b}r}, \\
\Gamma^{\bar{b}+} &: -\frac{1}{2}\epsilon_{\bar{b}}^{c_1}(\Omega_{\bar{a},c_1+} - iF_{\bar{a}c_1+c_2}^{c_2} + iF_{\bar{a}c_2+r}^r), \\
\Gamma^{\bar{q}_1\bar{q}_2} &: -\frac{i}{4}\epsilon^{c_1c_2}F_{\bar{a}c_1c_2\bar{q}_1\bar{q}_2} + \frac{1}{32}\epsilon_{\bar{q}_1\bar{q}_2}(G_{\bar{a}c}^c + G_{\bar{a}r}^r - G_{\bar{a}-+}), \\
\Gamma^{\bar{q}+} &: -\frac{i}{2}\epsilon^{c_1c_2}F_{\bar{a}c_1c_2\bar{q}+} + \frac{1}{16}\epsilon_{\bar{q}}^r G_{\bar{a}r+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}_1\bar{q}_2} &: \frac{1}{16}\epsilon_{\bar{b}_1\bar{b}_2}(\Omega_{\bar{a},\bar{q}_1\bar{q}_2} - iF_{\bar{a}\bar{q}_1\bar{q}_2}^c + iF_{\bar{a}\bar{q}_1\bar{q}_2-+}) - \frac{1}{32}\epsilon_{\bar{q}_1\bar{q}_2}G_{\bar{a}\bar{b}_1\bar{b}_2}, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}+} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2}(\Omega_{\bar{a},\bar{q}+} - iF_{\bar{a}\bar{q}+}^c + iF_{\bar{a}\bar{q}+}{}^r), \\
\Gamma^{\bar{b}\bar{q}_1\bar{q}_2+} &: -\frac{i}{4}\epsilon_{\bar{b}}^c F_{\bar{a}c\bar{q}_1\bar{q}_2+} - \frac{1}{32}\epsilon_{\bar{q}_1\bar{q}_2}G_{\bar{a}\bar{b}+}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.8}$$

The components along the p -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: -\frac{1}{2}\epsilon^{c_1c_2}(\Omega_{p,c_1c_2} + iF_{pc_1c_2}{}^r + iF_{pc_1c_2-+}), \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2} &: \frac{1}{4}\epsilon_{\bar{b}_1\bar{b}_2}(D_p - \frac{1}{2}\Omega_{p,c}^c + \frac{1}{2}\Omega_{p,r}^r + \frac{1}{2}\Omega_{p,-+} + \frac{i}{4}F_{pc_1}{}^{c_1}{}_{c_2}{}^{c_2} - \frac{i}{2}F_{pc}^c{}^r - \frac{i}{2}F_{p-+}^c + \frac{i}{2}F_{p-+}{}^r), \\
\Gamma^{\bar{b}\bar{q}} &: -\frac{1}{2}\epsilon_{\bar{b}}^{c_1}(\Omega_{p,c_1\bar{q}} + iF_{c_1p\bar{q}}{}^{c_2} - iF_{c_1p\bar{q}}{}^r - iF_{c_1p\bar{q}-+}) - \frac{1}{8}\epsilon_{\bar{q}}^r G_{\bar{b}pr}, \\
\Gamma^{\bar{b}+} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(\Omega_{p,c+} + iF_{ap+c}^c - F_{ap+r}^r),
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\bar{q}_1 \bar{q}_2} &: -\frac{i}{4} \epsilon^{c_1 c_2} F_{c_1 c_2 p \bar{q}_1 \bar{q}_2} - \frac{1}{32} \epsilon_{\bar{q}_1 \bar{q}_2} (G_{pc}{}^c - G_{pr}{}^r + G_{p-}{}^+), \\
\Gamma^{\bar{q}+} &: -\frac{i}{2} \epsilon^{c_1 c_2} F_{c_1 c_2 p \bar{q}+} + \frac{1}{8} \epsilon_{\bar{q}}{}^r G_{pr+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2} &: \frac{1}{16} \epsilon_{\bar{b}_1 \bar{b}_2} (\Omega_{p, \bar{q}_1 \bar{q}_2} - i F_{p \bar{q}_1 \bar{q}_2}{}^c + i F_{p \bar{q}_1 \bar{q}_2}{}^{+}) - \frac{1}{64} \epsilon_{\bar{q}_1 \bar{q}_2} G_{\bar{b}_1 \bar{b}_2 p}, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}+} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (\Omega_{p, \bar{q}+} - i F_{p \bar{q}+}{}^c + i F_{p \bar{q}+}{}^r), \\
\Gamma^{\bar{b} \bar{q}_1 \bar{q}_2 +} &: \frac{i}{4} \epsilon_{\bar{b}}{}^c F_{cp \bar{q}_1 \bar{q}_2 +} + \frac{1}{64} \epsilon_{\bar{q}_1 \bar{q}_2} G_{\bar{b} p +}, \\
\Gamma^{(5)} &: 0,
\end{aligned} \tag{B.9}$$

Similarly, the components along the \bar{p} -frame derivative are

$$\begin{aligned}
1 &: -\frac{1}{2} \epsilon^{c_1 c_2} (\Omega_{\bar{p}, c_1 c_2} + i F_{\bar{p} c_1 c_2}{}^r + i F_{\bar{p} c_1 c_2}{}^{+}) + \frac{1}{8} \epsilon^{r_1 r_2} (G_{\bar{p} r_1 r_2} - g_{\bar{p} q_1} (G_{q_2 c}{}^c + G_{q_2 r}{}^r + G_{q_2 -}{}^+)), \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2} &: \frac{1}{4} \epsilon_{\bar{b}_1 \bar{b}_2} (D_{\bar{p}} - \frac{1}{2} \Omega_{\bar{p}, c}{}^c + \frac{1}{2} \Omega_{\bar{p}, r}{}^r + \frac{1}{2} \Omega_{\bar{p}, -}{}^+ + \frac{i}{4} F_{\bar{p} c_1}{}^{c_1 c_2}{}^{c_2} - \frac{i}{2} F_{\bar{p} c}{}^c{}^r - \frac{i}{2} F_{\bar{p} -}{}^+{}^c \\
&\quad + \frac{i}{2} F_{\bar{p} -}{}^+{}^r) - \frac{1}{16} \epsilon_{\bar{p}}{}^r G_{\bar{b}_1 \bar{b}_2 r}, \\
\Gamma^{\bar{b} \bar{q}} &: -\frac{1}{2} \epsilon_{\bar{b}}{}^{c_1} (\Omega_{\bar{p}, c_1 \bar{q}} + i F_{c_1 \bar{p} \bar{q}}{}^{c_2} - i F_{c_1 \bar{p} \bar{q}}{}^{+}) + \frac{1}{16} \epsilon_{\bar{p}}{}^{r_1} (4 G_{\bar{b} \bar{q} r_1} - g_{\bar{q} r_1} (G_{\bar{b} c}{}^c + 3 G_{\bar{b} r_2}{}^{r_2} + G_{\bar{b} -}{}^+)), \\
\Gamma^{\bar{b}+} &: -\frac{1}{2} \epsilon_{\bar{b}}{}^c (\Omega_{\bar{p}, c+} + i F_{a \bar{p}+}{}^c - F_{a \bar{p}+}{}^r) + \frac{1}{8} \epsilon_{\bar{p}}{}^r G_{\bar{b} r+}, \\
\Gamma^{\bar{q}_1 \bar{q}_2} &: -\frac{1}{16} \epsilon_{\bar{q}_1 \bar{q}_2} (G_{\bar{p} c}{}^c - G_{\bar{p} r}{}^r + G_{\bar{p} -}{}^+), \\
\Gamma^{\bar{q}+} &: -\frac{i}{2} \epsilon^{c_1 c_2} F_{c_1 c_2 \bar{p} \bar{q}+} + \frac{1}{16} \epsilon_{\bar{q}}{}^{r_1} (4 G_{\bar{p} r_1+} - g_{\bar{p} r_1} (G_{+c}{}^c - G_{+r_2}{}^{r_2})), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2} &: \frac{1}{16} \epsilon_{\bar{b}_1 \bar{b}_2} \Omega_{\bar{p}, \bar{q}_1 \bar{q}_2} - \frac{1}{32} \epsilon_{\bar{q}_1 \bar{q}_2} G_{\bar{b}_1 \bar{b}_2 \bar{p}}, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}+} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (\Omega_{\bar{p}, \bar{q}+} - i F_{\bar{p} \bar{q}+}{}^c) + \frac{1}{32} \epsilon_{\bar{p} \bar{q}} G_{\bar{b}_1 \bar{b}_2 +}, \\
\Gamma^{\bar{b} \bar{q}_1 \bar{q}_2 +} &: \frac{1}{16} \epsilon_{\bar{q}_1 \bar{q}_2} G_{\bar{b} \bar{p}+}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.10}$$

The supercovariant derivative with $M = -$ gives

$$\begin{aligned}
1 &: -\frac{1}{2} \epsilon^{c_1 c_2} (\Omega_{-, c_1 c_2} + i F_{c_1 c_2 -}{}^r) + \frac{1}{4} \epsilon^{r_1 r_2} G_{r_1 r_2 -}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2} &: \frac{1}{4} \epsilon_{\bar{b}_1 \bar{b}_2} (D_- - \frac{1}{2} \Omega_{-, c}{}^c + \frac{1}{2} \Omega_{-, r}{}^r + \frac{1}{2} \Omega_{-, -}{}^+ + \frac{i}{4} F_{-c_1}{}^{c_1 c_2}{}^{c_2} - \frac{i}{2} F_{-c}{}^c{}^r + \frac{i}{4} F_{-r_1}{}^{r_1 r_2}{}^{r_2}), \\
\Gamma^{\bar{b} \bar{q}} &: -\frac{1}{2} \epsilon_{\bar{b}}{}^{c_1} (\Omega_{-, c \bar{q}} - i F_{c \bar{q} -}{}^c + i F_{c \bar{q} -}{}^r) + \frac{1}{4} \epsilon_{\bar{q}}{}^r G_{\bar{b} r -}, \\
\Gamma^{\bar{b}+} &: -\frac{1}{2} \epsilon_{\bar{b}}{}^{c_1} (\Omega_{-, c_1+} + i F_{c_1 -+}{}^{c_2} - i F_{c_1 -+}{}^r) + \frac{1}{16} \epsilon^{r_1 r_2} G_{\bar{b} r_1 r_2}, \\
\Gamma^{\bar{q}_1 \bar{q}_2} &: -\frac{i}{4} \epsilon^{c_1 c_2} F_{c_1 c_2 \bar{q}_1 \bar{q}_2 -} - \frac{1}{16} \epsilon_{\bar{q}_1 \bar{q}_2} (G_{-c}{}^c - G_{-r}{}^r), \\
\Gamma^{\bar{q}+} &: \frac{i}{2} \epsilon^{c_1 c_2} F_{c_1 c_2 \bar{q} -+} + \frac{1}{16} \epsilon_{\bar{q}}{}^{r_1} (G_{r_1 c}{}^c - G_{r_1 r_2}{}^{r_2} - 3 G_{r_1 -}{}^+), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2} &: \frac{1}{16} \epsilon_{\bar{b}_1 \bar{b}_2} (\Omega_{-, \bar{q}_1 \bar{q}_2} - i F_{\bar{q}_1 \bar{q}_2 -}{}^c) - \frac{1}{32} \epsilon_{\bar{q}_1 \bar{q}_2} G_{\bar{b}_1 \bar{b}_2 -}, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}+} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (\Omega_{-, \bar{q}+} + i F_{\bar{q} -+}{}^c - i F_{\bar{q} -+}{}^r) + \frac{1}{32} \epsilon_{\bar{q}}{}^r G_{\bar{b}_1 \bar{b}_2 r}, \\
\Gamma^{\bar{b} \bar{q}_1 \bar{q}_2 +} &: \frac{i}{4} \epsilon_{\bar{b}}{}^c F_{c \bar{q}_1 \bar{q}_2 -+} - \frac{1}{64} \epsilon_{\bar{q}_1 \bar{q}_2} (G_{\bar{b} c}{}^c - G_{\bar{b} r}{}^r - 3 G_{\bar{b} -}{}^+), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.11}$$

Finally, for $M = +$ we find

$$\begin{aligned}
1 &: -\frac{1}{2} \epsilon^{c_1 c_2} (\Omega_{+, c_1 c_2} + i F_{+c_1 c_2}{}^r) + \frac{1}{8} \epsilon^{r_1 r_2} G_{r_1 r_2 +}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2} &: \frac{1}{4} \epsilon_{\bar{b}_1 \bar{b}_2} (D_+ - \frac{1}{2} \Omega_{+, c}{}^c + \frac{1}{2} \Omega_{+, r}{}^r + \frac{1}{2} \Omega_{+, -}{}^+ + \frac{i}{4} F_{+c_1}{}^{c_1 c_2}{}^{c_2} - \frac{i}{2} F_{+c}{}^c{}^r + \frac{i}{4} F_{+r_1}{}^{r_1 r_2}{}^{r_2}),
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\bar{b}\bar{q}} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(\Omega_{+,c\bar{q}} - iF_{c\bar{q}+c}^c + iF_{c\bar{q}+r}^r) + \frac{1}{8}\epsilon_{\bar{q}}^r G_{\bar{b}r+}, \\
\Gamma^{\bar{b}+} &: -\frac{1}{2}\epsilon_{\bar{b}}^c \Omega_{+,c+}, \\
\Gamma^{\bar{q}_1\bar{q}_2} &: -\frac{i}{4}\epsilon^{c_1c_2} F_{c_1c_2\bar{q}_1\bar{q}_2+} - \frac{1}{32}\epsilon_{\bar{q}_1\bar{q}_2}(G_{+c}^c - G_{+r}^r), \\
\Gamma^{\bar{q}+} &: 0, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}_1\bar{q}_2} &: \frac{1}{16}\epsilon_{\bar{b}_1\bar{b}_2}(\Omega_{+,\bar{q}_1\bar{q}_2} - iF_{\bar{q}_1\bar{q}_2+c}^c) - \frac{1}{64}\epsilon_{\bar{q}_1\bar{q}_2} G_{\bar{b}_1\bar{b}_2+}, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}+} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2} \Omega_{+,\bar{q}+}, \\
\Gamma^{\bar{b}\bar{q}_1\bar{q}_2+} &: 0, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.12}$$

B.3 Killing spinor equations on e_{k5}

We now consider the basis elements $e_{k5} = \frac{1}{2}\Gamma^{\bar{k}}\Gamma^{+1}$ with $k \leq 4$. For this purpose we split up α into ρ and k , where ρ are the remaining three holomorphic indices: $\rho = (1, \dots, \hat{k}, \dots, 4)$. Thus, k is a single element of $\{1, 2, 3, 4\}$ and not an index that should be summed over. Furthermore we will use the three-dimensional Levi-Civita symbol defined by $\epsilon_{\rho_1 \dots \rho_3} = \epsilon_{k\rho_1 \dots \rho_3}$. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.8), we find

$$\begin{aligned}
1 &: 0, \\
\Gamma^{\bar{k}} &: \frac{1}{4}(G_{-k\bar{k}} - G_{-\gamma}^{\gamma}), \\
\Gamma^{\bar{\tau}} &: \frac{1}{2}G_{k\bar{\tau}-}, \\
\Gamma^{+} &: \frac{1}{4}(G_{k\gamma}^{\gamma} - G_{k-+}), \\
\Gamma^{(2)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{8}G_{\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: \frac{1}{8}(G_{\bar{\tau}k\bar{k}} - G_{\bar{\tau}\sigma}^{\sigma} + G_{\bar{\tau}-+}), \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3} &: -\frac{1}{12}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} P_{-}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma} P_{\sigma} + \frac{1}{8}G_{k\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{(4)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} P_{\bar{k}} - \frac{1}{48}G_{\bar{\tau}_1 \dots \bar{\tau}_3}.
\end{aligned} \tag{B.13}$$

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the k -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: -\Omega_{k,k-}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{k,\bar{\tau}-} + \frac{i}{2}F_{k\bar{\tau}-\sigma}^{\sigma} - \frac{1}{16}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2} G_{\sigma_1\sigma_2-}, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_k - \frac{1}{2}\Omega_{k,k\bar{k}} + \frac{1}{2}\Omega_{k,\sigma}^{\sigma} - \frac{1}{2}\Omega_{k,-+} + \frac{i}{4}F_{k\sigma_1}^{\sigma_1}\sigma_2^{\sigma_2} - \frac{i}{2}F_{k\sigma}^{\sigma} -) + \frac{1}{48}\epsilon^{\sigma_1 \dots \sigma_3} G_{\sigma_1 \dots \sigma_3}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma} G_{k\sigma-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{k,k\bar{\tau}} - \frac{1}{8}\epsilon_{\bar{\rho}}^{\sigma_1\sigma_2} G_{k\sigma_1\sigma_2}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3} &: \frac{i}{12}F_{k\bar{\tau}_1 \dots \bar{\tau}_3-} - \frac{1}{192}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3}(3G_{-k\bar{k}} + G_{-\sigma}^{\sigma}), \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{k,\bar{\tau}_1\bar{\tau}_2} + iF_{k\bar{\tau}_1\bar{\tau}_2\sigma}^{\sigma} - iF_{k\bar{\tau}_1\bar{\tau}_2-+}) + \frac{1}{64}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(3G_{\sigma_1 k\bar{k}} + G_{\sigma_1\sigma_2}^{\sigma_2} + G_{\sigma_1-+}), \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3+} &: -\frac{1}{96}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3}(G_{k\sigma}^{\sigma} + G_{k-+}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.14}$$

Along the \bar{k} -frame derivative of the supercovariant connection we find

$$\begin{aligned}
1 &: -\Omega_{\bar{k},k-} + iF_{k\bar{k}-\sigma}{}^\sigma, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{\bar{k},\bar{\tau}-} + \frac{i}{2}F_{\bar{k}\bar{\tau}-\sigma}{}^\sigma, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_{\bar{k}} - \frac{1}{2}\Omega_{\bar{k},k\bar{k}} + \frac{1}{2}\Omega_{\bar{k}\sigma}{}^\sigma - \frac{1}{2}\Omega_{\bar{k},-+} + \frac{i}{4}F_{\bar{k}\sigma_1}{}^{\sigma_1\sigma_2}{}^{\sigma_2} - \frac{i}{2}F_{\bar{k}\sigma}{}^\sigma{}_{-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: \frac{i}{2}F_{k\bar{k}\bar{\tau}_1\bar{\tau}_2-} - \frac{1}{16}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma G_{k\sigma-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{\bar{k},k\bar{\tau}} + \frac{i}{2}F_{k\bar{k}\bar{\tau}\sigma}{}^\sigma - \frac{i}{2}F_{k\bar{k}\bar{\tau}-+} - \frac{1}{16}\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} G_{\bar{k}\sigma_1\sigma_2}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{i}{12}F_{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{\bar{k},\bar{\tau}_1\bar{\tau}_2} + iF_{\bar{k}\bar{\tau}_1\bar{\tau}_2\sigma}{}^\sigma - iF_{\bar{k}\bar{\tau}_1\bar{\tau}_2-+}), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{i}{12}F_{k\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} - \frac{1}{192}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{\bar{k}\sigma}{}^\sigma + G_{\bar{k}-+}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.15}$$

The components along the ρ -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: -\Omega_{\rho,k-} + iF_{k\rho-\sigma}{}^\sigma, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{\rho,\bar{\tau}-} - \frac{i}{2}F_{\rho\bar{\tau}-k\bar{k}} + \frac{i}{2}F_{\rho\bar{\tau}-\sigma}{}^\sigma, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_{\rho} - \frac{1}{2}\Omega_{\rho,k\bar{k}} + \frac{1}{2}\Omega_{\rho,\sigma}{}^\sigma - \frac{1}{2}\Omega_{\rho,-+} - \frac{i}{2}F_{\rho k\bar{k}\sigma}{}^\sigma + \frac{i}{2}F_{\rho k\bar{k}-+} + \frac{i}{4}F_{\rho\sigma_1}{}^{\sigma_1\sigma_2}{}^{\sigma_2} - \frac{i}{2}F_{\rho\sigma}{}^\sigma{}_{-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: \frac{i}{2}F_{k\rho\bar{\tau}_1\bar{\tau}_2-} - \frac{1}{16}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma G_{\rho\sigma-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{\rho,k\bar{\tau}} + \frac{i}{2}F_{k\rho\bar{\tau}\sigma}{}^\sigma - \frac{i}{2}F_{k\rho\bar{\tau}-+} - \frac{1}{16}\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} G_{\rho\sigma_1\sigma_2}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{i}{12}F_{\rho\bar{\tau}_1\cdots\bar{\tau}_3-} + \frac{1}{96}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3} G_{\bar{k}\rho-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{\rho,\bar{\tau}_1\bar{\tau}_2} - iF_{k\bar{k}\rho\bar{\tau}_1\bar{\tau}_2} + iF_{\rho\bar{\tau}_1\bar{\tau}_2\sigma}{}^\sigma - iF_{\rho\bar{\tau}_1\bar{\tau}_2-+}) - \frac{1}{32}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma G_{\bar{k}\rho\sigma}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{i}{12}F_{k\rho\bar{\tau}_1\cdots\bar{\tau}_3} + \frac{1}{192}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{\rho k\bar{k}} - G_{\rho\sigma}{}^\sigma - G_{\rho-+}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.16}$$

Similarly, the components along the $\bar{\rho}$ -frame derivative are

$$\begin{aligned}
1 &: -\Omega_{\bar{\rho},k-} + iF_{k\bar{\rho}-\sigma}{}^\sigma - \frac{1}{8}\epsilon_{\bar{\rho}}{}^{\sigma_1\sigma_2} G_{\sigma_1\sigma_2-}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{\bar{\rho},\bar{\tau}-} - \frac{i}{2}F_{\bar{\rho}\bar{\tau}-k\bar{k}} + \frac{i}{2}F_{\bar{\rho}\bar{\tau}-\sigma}{}^\sigma - \frac{1}{8}\epsilon_{\bar{\rho}\bar{\tau}}{}^\sigma G_{\bar{k}\sigma-}, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_{\bar{\rho}} - \frac{1}{2}\Omega_{\bar{\rho},k\bar{k}} + \frac{1}{2}\Omega_{\bar{\rho},\sigma}{}^\sigma - \frac{1}{2}\Omega_{\bar{\rho},-+} - \frac{i}{2}F_{\bar{\rho}k\bar{k}\sigma}{}^\sigma + \frac{i}{2}F_{\bar{\rho}k\bar{k}-+} + \frac{i}{4}F_{\bar{\rho}\sigma_1}{}^{\sigma_1\sigma_2}{}^{\sigma_2} \\
&\quad - \frac{i}{2}F_{\bar{\rho}\sigma}{}^\sigma{}_{-+}) - \frac{1}{16}\epsilon_{\bar{\rho}}{}^{\sigma_1\sigma_2} G_{\bar{k}\sigma_1\sigma_2}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: \frac{i}{2}F_{k\bar{\rho}\bar{\tau}_1\bar{\tau}_2-} + \frac{1}{32}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(g_{\bar{\rho}\sigma_1}G_{-k\bar{k}} - g_{\bar{\rho}\sigma_1}G_{-\sigma_2}{}^{\sigma_2} - 4G_{\bar{\rho}\sigma_1-}), \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{\bar{\rho},k\bar{\tau}} + \frac{i}{2}F_{k\bar{\rho}\bar{\tau}\sigma}{}^\sigma - \frac{i}{2}F_{k\bar{\rho}\bar{\tau}-+} - \frac{1}{16}\epsilon_{\bar{\rho}}{}^{\sigma_1\sigma_2}(2G_{\bar{\tau}\sigma_1\sigma_2} + g_{\bar{\tau}\sigma_1}(G_{\sigma_2 k\bar{k}} + 3G_{\sigma_2\sigma_3}{}^{\sigma_3} - G_{\sigma_2-+})), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: +\frac{1}{48}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3} G_{\bar{k}\bar{\rho}-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{\bar{\rho},\bar{\tau}_1\bar{\tau}_2} - iF_{k\bar{k}\bar{\rho}\bar{\tau}_1\bar{\tau}_2} - iF_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2-+}) - \frac{1}{64}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(4G_{\bar{k}\bar{\rho}\sigma_1} + g_{\bar{\rho}\sigma_1}(G_{\bar{k}\sigma_2}{}^{\sigma_2} + G_{\bar{k}-+})), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{96}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{\bar{\rho}k\bar{k}} - G_{\bar{\rho}\sigma}{}^\sigma - G_{\bar{\rho}-+}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.17}$$

The supercovariant derivative with $M = -$ gives

$$1 : -\Omega_{-,k-},$$

$$\begin{aligned}
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{-,\bar{\tau}-}, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_- - \frac{1}{2}\Omega_{-,k\bar{k}} + \frac{1}{2}\Omega_{-,\sigma}{}^\sigma - \frac{1}{2}\Omega_{-,-+} - \frac{i}{2}F_{-\sigma}{}^\sigma{}_{k\bar{k}} + \frac{i}{4}F_{-\sigma_1}{}^{\sigma_1}{}_{\sigma_2}{}^{\sigma_2}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: 0, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{-,k\bar{\tau}} - \frac{i}{2}F_{k\bar{\tau}\sigma}{}^\sigma - \frac{1}{16}\epsilon_{\bar{\rho}}{}^{\sigma_1\sigma_2}G_{-\sigma_1\sigma_2}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{-,\bar{\tau}_1\bar{\tau}_2} + iF_{-\bar{\tau}_1\bar{\tau}_2\sigma}{}^\sigma - iF_{-\bar{\tau}_1\bar{\tau}_2k\bar{k}}) + \frac{1}{32}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma G_{\bar{k}\sigma-}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: -\frac{i}{12}F_{k\bar{\tau}_1\cdots\bar{\tau}_3-} + \frac{1}{192}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{-k\bar{k}} - G_{-\sigma}{}^\sigma), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.18}$$

Finally, for $M = +$ we find

$$\begin{aligned}
1 &: -\Omega_{+,k-} - iF_{k-+}{}^\sigma + \frac{1}{24}\epsilon^{\sigma_1\cdots\sigma_3}G_{\sigma_1\cdots\sigma_3}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{2}\Omega_{+,\bar{\tau}-} - \frac{i}{2}F_{\bar{\tau}-+k\bar{k}} + \frac{i}{2}F_{\bar{\tau}-+}{}^\sigma + \frac{1}{16}\epsilon_{\bar{\tau}}{}^{\sigma_2\sigma_2}G_{\bar{k}\sigma_1\sigma_2}, \\
\Gamma^{\bar{k}+} &: \frac{1}{2}(D_+ - \frac{1}{2}\Omega_{+,k\bar{k}} + \frac{1}{2}\Omega_{+,\sigma}{}^\sigma - \frac{1}{2}\Omega_{+,-+} - \frac{i}{2}F_{+k\bar{k}\sigma}{}^\sigma + \frac{i}{4}F_{+\sigma_1}{}^{\sigma_1}{}_{\sigma_2}{}^{\sigma_2}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{i}{2}F_{k\bar{\tau}_1\bar{\tau}_2-+} - \frac{1}{32}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(G_{\sigma_1k\bar{k}} - G_{\sigma_1\sigma_2}{}^{\sigma_2} + 3G_{\sigma_1-+}), \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\Omega_{+,k\bar{\tau}} - \frac{i}{2}F_{k\bar{\tau}+}{}^\sigma - \frac{1}{8}\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2}G_{+\sigma_1\sigma_2}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{i}{12}F_{\bar{\tau}_1\cdots\bar{\tau}_3-+} + \frac{1}{192}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{\bar{k}\sigma}{}^\sigma - 3G_{\bar{k}-+}), \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}(\Omega_{+,\bar{\tau}_1\bar{\tau}_2} - iF_{k\bar{k}\bar{\tau}_1\bar{\tau}_2+} + iF_{\bar{\tau}_1\bar{\tau}_2+}{}^\sigma) + \frac{1}{16}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma G_{\bar{k}\sigma+}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: -\frac{i}{12}F_{k\bar{\tau}_1\cdots\bar{\tau}_3+} + \frac{1}{96}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(G_{+k\bar{k}} - G_{+\sigma}{}^\sigma), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.19}$$

B.4 Killing spinor equations on e_{1234}

The next spinor basis element we consider is e_{1234} . Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.8), we find

$$\begin{aligned}
1 &: 0, \\
\Gamma^{\bar{\beta}} &: P_{\bar{\beta}} + \frac{1}{12}\epsilon_{\bar{\beta}}{}^{\gamma_1\cdots\gamma_3}G_{\gamma_1\cdots\gamma_3}, \\
\Gamma^+ &: P_+, \\
\Gamma^{(2)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: \frac{1}{48}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^\gamma(G_{\gamma\delta}{}^\delta - G_{\gamma-+}), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: -\frac{1}{16}\epsilon_{\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1\gamma_2}G_{\gamma_1\gamma_2+}, \\
\Gamma^{(4)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: -\frac{1}{384}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}G_{+\gamma}{}^\gamma.
\end{aligned} \tag{B.20}$$

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the α -frame derivative of the supercovariant connection are

$$1 : \frac{1}{4}(G_{\alpha\gamma}{}^\gamma + G_{\alpha-+}),$$

$$\begin{aligned}
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: -\frac{1}{8}\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}(\Omega_{\alpha,\gamma_1\gamma_2} - iF_{\alpha\gamma_1\gamma_2}\delta^\delta + iF_{\alpha\gamma_1\gamma_2-+}) + \frac{1}{8}G_{\alpha\bar{\beta}_1\bar{\beta}_2} - \frac{1}{16}g_{\alpha[\bar{\beta}_1}(G_{\bar{\beta}_2]\gamma}^\gamma + G_{\bar{\beta}_2]-+}), \\
\Gamma^{\bar{\beta}+} &: \frac{i}{6}\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}F_{\alpha\gamma_1\cdots\gamma_3+} + \frac{1}{4}G_{\alpha\bar{\beta}+} - \frac{1}{16}g_{\alpha\bar{\beta}}G_{+\gamma}^\gamma, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(D_\alpha - \frac{1}{2}\Omega_{\alpha,\gamma}^\gamma + \frac{1}{2}\Omega_{\alpha,-+} + \frac{i}{4}F_{\alpha\gamma_1}^{\gamma_1\gamma_2} - \frac{i}{2}F_{\alpha\gamma_1}^{\gamma_1-+}) - \frac{1}{96}g_{\alpha[\bar{\beta}_1}G_{\bar{\beta}_2\cdots\bar{\beta}_4]}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: -\frac{1}{24}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^\gamma(\Omega_{\alpha,\gamma+} - iF_{\alpha\gamma+\delta}^\delta) - \frac{1}{32}g_{\alpha[\bar{\beta}_1}G_{\bar{\beta}_2\bar{\beta}_3]+}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.21}$$

Along the $\bar{\alpha}$ -frame derivative of the supercovariant connection we find

$$\begin{aligned}
1 &: \frac{i}{12}\epsilon^{\beta_1\cdots\beta_4}F_{\bar{\alpha}\beta_1\cdots\beta_4} + \frac{1}{8}(G_{\bar{\alpha}\gamma}^\gamma + G_{\bar{\alpha}-+}), \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: -\frac{1}{8}\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}(\Omega_{\bar{\alpha},\gamma_1\gamma_2} - iF_{\bar{\alpha}\gamma_1\gamma_2}\delta^\delta + iF_{\bar{\alpha}\gamma_1\gamma_2-+}) + \frac{1}{16}G_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2}, \\
\Gamma^{\bar{\beta}+} &: \frac{i}{6}\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}F_{\bar{\alpha}\gamma_1\cdots\gamma_3+} + \frac{1}{8}G_{\bar{\alpha}\bar{\beta}+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(D_{\bar{\alpha}} - \frac{1}{2}\Omega_{\bar{\alpha},\gamma}^\gamma + \frac{1}{2}\Omega_{\bar{\alpha},-+} + \frac{i}{4}F_{\bar{\alpha}\gamma_1}^{\gamma_1\gamma_2} - \frac{i}{2}F_{\bar{\alpha}\gamma_1}^{\gamma_1-+}), \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: -\frac{1}{24}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^\gamma(\Omega_{\bar{\alpha},\gamma+} - iF_{\bar{\alpha}\gamma+\delta}^\delta), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.22}$$

The supercovariant derivative with $M = -$ gives

$$\begin{aligned}
1 &: \frac{i}{12}\epsilon^{\beta_1\cdots\beta_4}F_{-\beta_1\cdots\beta_4} + \frac{1}{4}G_{-\gamma}^\gamma, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: -\frac{1}{8}\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}(\Omega_{-, \gamma_1\gamma_2} - iF_{-\gamma_1\gamma_2}\delta^\delta) + \frac{1}{8}G_{-\bar{\beta}_1\bar{\beta}_2}, \\
\Gamma^{\bar{\beta}+} &: -\frac{i}{6}\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}F_{\gamma_1\cdots\gamma_3-+} + \frac{1}{16}(G_{\bar{\beta}\gamma}^\gamma - 3G_{\bar{\beta}-+}), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(D_- - \frac{1}{2}\Omega_{-, \gamma}^\gamma + \frac{1}{2}\Omega_{-, -+} + \frac{i}{4}F_{-\gamma_1}^{\gamma_1\gamma_2}), \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: -\frac{1}{24}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^\gamma(\Omega_{-, \gamma+} + iF_{-\gamma+\delta}^\delta) + \frac{1}{96}G_{\bar{\beta}_1\cdots\bar{\beta}_3}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.23}$$

Finally, for $M = +$ we find

$$\begin{aligned}
1 &: \frac{i}{12}\epsilon^{\beta_1\cdots\beta_4}F_{\beta_1\cdots\beta_4+} + \frac{1}{8}G_{+\gamma}^\gamma, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: -\frac{1}{8}\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}(\Omega_{+, \gamma_1\gamma_2} - iF_{\gamma_1\gamma_2+\delta}^\delta) + \frac{1}{16}G_{\bar{\beta}_1\bar{\beta}_2+}, \\
\Gamma^{\bar{\beta}+} &: 0, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(D_+ - \frac{1}{2}\Omega_{+, \gamma}^\gamma + \frac{1}{2}\Omega_{+, -+} + \frac{i}{4}F_{+\gamma_1}^{\gamma_1\gamma_2}), \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: -\frac{1}{24}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^\gamma\Omega_{+, \gamma+}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.24}$$

B.5 Killing spinor equations on $e_{i_1\cdots i_3 5}$

We now consider the basis elements $e_{i_1\cdots i_3 5} = \frac{1}{4}\Gamma^{i_1\cdots i_3}\Gamma^{+1}$ with $i_1, i_2, i_3 \leq 4$. For this purpose we split up α into ρ and k , where $\rho = (i_1, \dots, i_3)$ and k is the missing fourth

holomorphic coordinate (and is not an index that should be summed over). Again we will use the three-dimensional Levi-Civita symbol defined by $\epsilon_{\rho_1 \dots \rho_3} = \epsilon_{k\rho_1 \dots \rho_3}$. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.8), we find

$$\begin{aligned}
1 &: 0, \\
\Gamma^{\bar{k}} &: -P_-, \\
\Gamma^{\bar{\tau}} &: \frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}G_{\sigma_1\sigma_2-}, \\
\Gamma^+ &: P_k - \frac{1}{12}\epsilon^{\sigma_1\dots\sigma_3}G_{\sigma_1\dots\sigma_3}, \\
\Gamma^{(2)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}G_{\bar{k}\sigma-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{2}P_{\bar{\tau}} - \frac{1}{8}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}G_{\bar{k}\sigma_1\sigma_2}, \\
\Gamma^{\bar{\tau}_1\dots\bar{\tau}_3} &: -\frac{1}{48}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}(G_{-k\bar{k}} - G_{-\sigma}^{\sigma}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{16}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(G_{\sigma_1 k\bar{k}} - G_{\sigma_1\sigma_2}^{\sigma_2} - G_{\sigma_2-+}), \\
\Gamma^{(4)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\dots\bar{\tau}_3+} &: -\frac{1}{96}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}(G_{\bar{k}\sigma}^{\sigma} + G_{\bar{k}-+}).
\end{aligned} \tag{B.25}$$

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the k -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: \frac{i}{3}\epsilon^{\sigma_1\dots\sigma_3}F_{k\sigma_1\dots\sigma_3-}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{i}{2}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}F_{\bar{k}\bar{\sigma}_1\sigma_2-} + \frac{1}{8}G_{k\bar{\tau}-}, \\
\Gamma^{\bar{k}+} &: -\frac{i}{6}\epsilon^{\sigma_1\dots\sigma_3}F_{\bar{k}\bar{\sigma}_1\dots\sigma_3} + \frac{1}{16}(G_{k\sigma}^{\sigma} - G_{k-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{k,\sigma_1-} - iF_{k\sigma_1-\sigma_2}^{\sigma_2}), \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{k,\sigma_1\sigma_2} - iF_{k\sigma_1\sigma_2\sigma_3}^{\sigma_3} - iF_{k\sigma_1\sigma_2-+}), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\dots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}(\Omega_{k,\bar{k}-} - iF_{\bar{k}\bar{k}-\sigma}^{\sigma}), \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{k,\bar{k}\sigma_1} - iF_{\bar{k}\bar{\sigma}_1\sigma_2}^{\sigma_2} - iF_{\bar{k}\bar{\sigma}_1-+}) + \frac{1}{32}G_{k\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{\tau}_1\dots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}(D_k + \frac{1}{2}\Omega_{k,\bar{k}\bar{k}} - \frac{1}{2}\Omega_{k,\sigma}^{\sigma} - \frac{1}{2}\Omega_{k,-+} + \frac{i}{4}F_{k\sigma_1}^{\sigma_1}\sigma_2^{\sigma_2} + \frac{i}{2}F_{k\sigma}^{\sigma}-+), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.26}$$

Along the \bar{k} -frame derivative of the supercovariant connection we find

$$\begin{aligned}
1 &: \frac{i}{3}\epsilon^{\sigma_1\dots\sigma_3}F_{\bar{k}\sigma_1\dots\sigma_3-} + \frac{1}{8}(3G_{-k\bar{k}} + G_{-\sigma}^{\sigma}), \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{1}{4}G_{\bar{k}\bar{\tau}-}, \\
\Gamma^{\bar{k}+} &: \frac{1}{8}(G_{\bar{k}\sigma}^{\sigma} - G_{\bar{k}-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{\bar{k},\sigma_1-} - iF_{\bar{k}\sigma_1-\sigma_2}^{\sigma_2}) + \frac{1}{16}G_{\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{\bar{k},\sigma_1\sigma_2} - iF_{\bar{k}\sigma_1\sigma_2\sigma_3}^{\sigma_3} - iF_{\bar{k}\sigma_1\sigma_2-+}) + \frac{1}{16}(3G_{\bar{\tau}k\bar{k}} + G_{\bar{\tau}\sigma}^{\sigma} - G_{\bar{\tau}-+}), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\dots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}\Omega_{\bar{k},\bar{k}-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}\Omega_{\bar{k},\bar{k}\sigma} + \frac{1}{16}G_{\bar{k}\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{\tau}_1\dots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\dots\bar{\tau}_3}(D_{\bar{k}} + \frac{1}{2}\Omega_{\bar{k},\bar{k}\bar{k}} - \frac{1}{2}\Omega_{\bar{k},\sigma}^{\sigma} - \frac{1}{2}\Omega_{\bar{k},-+} + \frac{i}{4}F_{\bar{k}\sigma_1}^{\sigma_1}\sigma_2^{\sigma_2} + \frac{i}{2}F_{\bar{k}\sigma}^{\sigma}-+) + \frac{1}{96}G_{\bar{\tau}_1\dots\bar{\tau}_3}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.27}$$

The components along the ρ -frame derivative of the supercovariant connection are

$$\begin{aligned}
1 &: \frac{1}{2}G_{k\rho-}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: -\frac{i}{2}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}F_{\bar{k}\rho\sigma_1\sigma_2-} + \frac{1}{4}G_{\rho\bar{\tau}-} + \frac{1}{16}g_{\rho\bar{\tau}}(G_{-k\bar{k}} - G_{-\sigma}{}^{\sigma}), \\
\Gamma^{\bar{k}+} &: -\frac{1}{8}(G_{\rho k\bar{k}} - G_{k\sigma}{}^{\sigma} + G_{\rho-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{\rho,\sigma_1-} + iF_{\rho\sigma_1-k\bar{k}} - iF_{\rho\sigma_1-\sigma_2}{}^{\sigma_2}) + \frac{1}{8}g_{\rho[\bar{\tau}_1}G_{\bar{\tau}_2]k-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{\rho,\sigma_1\sigma_2} + iF_{\rho\sigma_1\sigma_2 k\bar{k}} - iF_{\rho\sigma_1\sigma_2-+}) + \frac{1}{4}G_{k\rho\bar{\tau}} + \frac{1}{16}g_{\rho\bar{\tau}}(-G_{k\sigma}{}^{\sigma} + G_{k-+}), \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(\Omega_{\rho,\bar{k}-} + iF_{\bar{k}\rho-\sigma}{}^{\sigma}) - \frac{1}{32}g_{\rho[\bar{\tau}_1}G_{\bar{\tau}_2\bar{\tau}_3]-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{\rho,\bar{k}\sigma_1} + iF_{\bar{k}\rho\sigma_1\sigma_2}{}^{\sigma_2} + iF_{\bar{k}\rho\sigma_1-+}) + \frac{1}{16}G_{\rho\bar{\tau}_1\bar{\tau}_2} \\
&\quad + \frac{1}{32}g_{\rho[\bar{\tau}_1}(G_{\bar{\tau}_2]k\bar{k}} - G_{\bar{\tau}_2]\sigma_2}{}^{\sigma_2} + G_{\bar{\tau}_2]-+}), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(D_{\rho} + \frac{1}{2}\Omega_{\rho,k\bar{k}} - \frac{1}{2}\Omega_{\rho,\sigma}{}^{\sigma} - \frac{1}{2}\Omega_{\rho,-+} - \frac{i}{2}F_{\rho k\bar{k}\sigma}{}^{\sigma} - \frac{i}{2}F_{\rho k\bar{k}-+} \\
&\quad + \frac{i}{4}F_{\rho\sigma_1}{}^{\sigma_1}{}^{\sigma_2}{}^{\sigma_2} + \frac{i}{2}F_{\rho\sigma}{}^{\sigma}{}_{-+}) - \frac{1}{32}g_{\rho[\bar{\tau}_1}G_{\bar{\tau}_2\bar{\tau}_3]k}, \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.28}$$

Similarly, the components along the $\bar{\rho}$ -frame derivative are

$$\begin{aligned}
1 &: \frac{i}{3}\epsilon^{\sigma_1\cdots\sigma_3}F_{\bar{\rho}\sigma_1\cdots\sigma_3-} + \frac{1}{4}G_{k\bar{\rho}-}, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: -\frac{i}{2}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}F_{\bar{k}\bar{\rho}\sigma_1\sigma_2-} + \frac{1}{8}G_{\bar{\rho}\bar{\tau}-}, \\
\Gamma^{\bar{k}+} &: \frac{i}{6}\epsilon^{\sigma_1\cdots\sigma_3}F_{\bar{k}\bar{\rho}\sigma_1\cdots\sigma_3} - \frac{1}{16}(G_{\bar{\rho}k\bar{k}} - G_{\bar{k}\sigma}{}^{\sigma} + G_{\bar{\rho}-+}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{\bar{\rho},\sigma_1-} + iF_{\bar{\rho}\sigma_1-k\bar{k}} - iF_{\bar{\rho}\sigma_1-\sigma_2}{}^{\sigma_2}), \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{\bar{\rho},\sigma_1\sigma_2} + iF_{\bar{\rho}\sigma_1\sigma_2 k\bar{k}} - iF_{\bar{\rho}\sigma_1\sigma_2\sigma_3}{}^{\sigma_3} - iF_{\bar{\rho}\sigma_1\sigma_2-+}) + \frac{1}{8}G_{k\bar{\rho}\bar{\tau}}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(\Omega_{\bar{\rho},\bar{k}-} + iF_{\bar{k}\bar{\rho}-\sigma}{}^{\sigma}), \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{\bar{\rho},\bar{k}\sigma_1} + iF_{\bar{k}\bar{\rho}\sigma_1\sigma_2}{}^{\sigma_2} + iF_{\bar{k}\bar{\rho}\sigma_1-+}) + \frac{1}{32}G_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(D_{\bar{\rho}} + \frac{1}{2}\Omega_{\bar{\rho},k\bar{k}} - \frac{1}{2}\Omega_{\bar{\rho},\sigma}{}^{\sigma} - \frac{1}{2}\Omega_{\bar{\rho},-+} \\
&\quad - \frac{i}{2}F_{\bar{\rho}k\bar{k}\sigma}{}^{\sigma} - \frac{i}{2}F_{\bar{\rho}k\bar{k}-+} + \frac{i}{4}F_{\bar{\rho}\sigma_1}{}^{\sigma_1}{}^{\sigma_2}{}^{\sigma_2} + \frac{i}{2}F_{\bar{\rho}\sigma}{}^{\sigma}{}_{-+}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.29}$$

The supercovariant derivative with $M = -$ gives

$$\begin{aligned}
1 &: 0, \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: 0, \\
\Gamma^{\bar{k}+} &: -\frac{i}{6}\epsilon^{\sigma_1\cdots\sigma_3}F_{\bar{k}\sigma_1\cdots\sigma_3-} - \frac{1}{16}(G_{-k\bar{k}} - G_{-\sigma}{}^{\sigma}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}\Omega_{-,\sigma-}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{-,\sigma_1\sigma_2} + iF_{-\sigma_1\sigma_2 k\bar{k}} - iF_{-\sigma_1\sigma_2\sigma_3}{}^{\sigma_3}) - \frac{1}{8}G_{k\bar{\tau}-}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}\Omega_{-,\bar{k}-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{-,\bar{k}\sigma_1} - iF_{\bar{k}\sigma_1-\sigma_2}{}^{\sigma_2}) + \frac{1}{32}G_{-\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(D_{-} + \frac{1}{2}\Omega_{-,\bar{k}\bar{k}} - \frac{1}{2}\Omega_{-,\sigma}{}^{\sigma} - \frac{1}{2}\Omega_{-,-+} - \frac{i}{2}F_{-\sigma}{}^{\sigma}{}_{k\bar{k}} + \frac{i}{4}F_{-\sigma_1}{}^{\sigma_1}{}^{\sigma_2}{}^{\sigma_2}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.30}$$

Finally, for $M = +$ we find

$$\begin{aligned}
1 &: \frac{i}{3}\epsilon^{\sigma_1\cdots\sigma_3}F_{\sigma_1\cdots\sigma_3-+} - \frac{1}{8}(G_{k\sigma}{}^\sigma + 3G_{k-+}), \\
\Gamma^{(1)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}} &: \frac{i}{2}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}F_{\bar{k}\sigma_1\sigma_2-+} - \frac{1}{16}(G_{\bar{\tau}k\bar{k}} - G_{\bar{\tau}\sigma}{}^\sigma - 3G_{\bar{\tau}-+}), \\
\Gamma^{\bar{k}+} &: -\frac{i}{6}\epsilon^{\sigma_1\cdots\sigma_3}F_{\bar{k}\sigma_1\cdots\sigma_3+} - \frac{1}{8}(G_{+k\bar{k}} - G_{+\sigma}{}^\sigma), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{+, \sigma_1-} + iF_{\sigma_1 k\bar{k}-+} - iF_{\sigma_1-+ \sigma_2}{}^{\sigma_2}) - \frac{1}{16}G_{k\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{4}\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(\Omega_{+, \sigma_1\sigma_2} + iF_{+\sigma_1\sigma_2 k\bar{k}} - iF_{+\sigma_1\sigma_2\sigma_3}{}^{\sigma_3}) - \frac{1}{4}G_{k\bar{\tau}+}, \\
\Gamma^{(3)} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(\Omega_{+, \bar{k}-} - iF_{\bar{k}-+ \sigma}{}^\sigma) + \frac{1}{96}G_{\bar{\tau}_1\cdots\bar{\tau}_3}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(\Omega_{+, \bar{k}\sigma_1} - iF_{\bar{k}\sigma_1+ \sigma_2}{}^{\sigma_2}) + \frac{1}{16}G_{\bar{\tau}_1\bar{\tau}_2+}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(D_{+} + \frac{1}{2}\Omega_{+, k\bar{k}} - \frac{1}{2}\Omega_{+, \sigma}{}^\sigma - \frac{1}{2}\Omega_{+, -+} - \frac{i}{2}F_{k\bar{k}+ \sigma}{}^\sigma + \frac{i}{4}F_{+\sigma_1}{}^{\sigma_1}\sigma_2{}^{\sigma_2}), \\
\Gamma^{(5)} &: 0.
\end{aligned} \tag{B.31}$$

C Integrability conditions

C.1 Integrability conditions on 1

The expressions for the integrability condition \mathcal{I} on the basis element 1 read

$$\begin{aligned}
1 &: LG_\gamma{}^\gamma + LG_{-+} + 12BG_{\gamma_1}{}^{\gamma_1}\gamma_2{}^{\gamma_2} + 24BG_\gamma{}^\gamma{}_{-+}, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: \frac{1}{2}LG_{\bar{\beta}_1\bar{\beta}_2} + 12BG_{\bar{\beta}_1\bar{\beta}_2}\gamma{}^\gamma + 12BG_{\bar{\beta}_1\bar{\beta}_2-+} - \frac{1}{2}\epsilon_{\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1\gamma_2}BP_{\gamma_1\gamma_2}, \\
\Gamma^{\bar{\beta}+} &: LG_{\bar{\beta}+} + 24BG_{\bar{\beta}\gamma}{}^\gamma, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: BG_{\bar{\beta}_1\cdots\bar{\beta}_4} + \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(LP - 2BP_\gamma{}^\gamma + 2BP_{-+}), \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: 4BG_{\bar{\beta}_1\cdots\bar{\beta}_3+} - \frac{1}{6}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^\gamma BP_{\gamma+}.
\end{aligned} \tag{C.1}$$

The integrability conditions \mathcal{I}_α are

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: \frac{1}{2}E_{\alpha\bar{\beta}} - 6iLF_{\alpha\bar{\beta}\gamma}{}^\gamma - 6iLF_{\alpha\bar{\beta}-+} + 8\epsilon_{\bar{\beta}}{}^{\gamma_1\cdots\gamma_3}BG_{\alpha\gamma_1\cdots\gamma_3}, \\
\Gamma^+ &: \frac{1}{2}E_{\alpha+} - 6iLF_{\alpha+ \gamma}{}^\gamma, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -iLF_{\alpha\bar{\beta}_1\cdots\bar{\beta}_3} + \frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^\gamma(LG_{\alpha\gamma} + 24BG_{\alpha\gamma_1\gamma_2}{}^{\gamma_2} - 24BG_{\alpha\gamma_1-+}), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: -3iLF_{\alpha\bar{\beta}_1\bar{\beta}_2+} - 6\epsilon_{\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1\gamma_2}BG_{\alpha\gamma_1\gamma_2+}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: -\frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(LG_{\alpha+} + 24BG_{\alpha+ \gamma}{}^\gamma).
\end{aligned} \tag{C.2}$$

The integrability conditions $\mathcal{I}_{\bar{\alpha}}$ read

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: \frac{1}{2}E_{\bar{\alpha}\bar{\beta}} - 6i(LF_{\bar{\alpha}\bar{\beta}\gamma}{}^\gamma + LF_{\bar{\alpha}\bar{\beta}-+}) - 8\epsilon_{\bar{\beta}}{}^{\gamma_1\cdots\gamma_3}BG_{\bar{\alpha}\gamma_1\cdots\gamma_3} + \\
&\quad + 24\epsilon_{\bar{\alpha}\bar{\beta}}{}^{\gamma_1\gamma_2}(BG_{\gamma_1\gamma_2\gamma_3}{}^{\gamma_3} - BG_{\gamma_1\gamma_2-+}), \\
\Gamma^+ &: \frac{1}{2}E_{\bar{\alpha}+} - 6iLF_{\bar{\alpha}+ \gamma}{}^\gamma - 16\epsilon_{\bar{\alpha}}{}^{\gamma_1\cdots\gamma_3}BG_{+\gamma_1\cdots\gamma_3}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -iLF_{\bar{\alpha}\bar{\beta}_1\cdots\bar{\beta}_3} + \frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^{\gamma_1}(LG_{\bar{\alpha}\gamma_1} - 24BG_{\bar{\alpha}\gamma_1\gamma_2}{}^{\gamma_2} + 24BG_{\bar{\alpha}\gamma_1-+}) + \\
&\quad + \epsilon_{\bar{\alpha}\bar{\beta}_1\cdots\bar{\beta}_3}(BG_{\gamma_1}{}^{\gamma_1}\gamma_2{}^{\gamma_2} - 2BG_{\gamma_1}{}^{\gamma_1}{}_{-+}), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: -3iLF_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2+} + 12\epsilon_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1}BG_{\gamma_1+ \gamma_2}{}^{\gamma_2} + 6\epsilon_{\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1\gamma_2}BG_{\bar{\alpha}\gamma_1\gamma_2+}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: -\frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(LG_{\bar{\alpha}+} - 24BG_{\bar{\alpha}+ \gamma}{}^\gamma).
\end{aligned} \tag{C.3}$$

Similarly, \mathcal{I}_A with $A = -$ is given by

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: \frac{1}{2}E_{-\bar{\beta}} - 6iLF_{-\bar{\beta}\gamma}{}^{\gamma} + 8\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}BG_{\gamma_1\cdots\gamma_3-}, \\
\Gamma^+ &: \frac{1}{2}E_{-+} - 6iLF_{-+\gamma}{}^{\gamma} + 4\epsilon^{\gamma_1\cdots\gamma_4}BG_{\gamma_1\cdots\gamma_4}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -iLF_{-\bar{\beta}_1\cdots\bar{\beta}_3} - \frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^{\gamma}(LG_{\gamma-} - 24BG_{\gamma_1\gamma_2}{}^{\gamma_3-}), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: -3iLF_{-\bar{\beta}_1\bar{\beta}_2+} + 6\epsilon_{\bar{\beta}_1\bar{\beta}_2}{}^{\gamma_1\gamma_2}BG_{\gamma_1\gamma_2\gamma_3}{}^{\gamma_3}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: -\frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(LG_{-+} - 12BG_{\gamma_1}{}^{\gamma_1}\gamma_2{}^{\gamma_2}).
\end{aligned} \tag{C.4}$$

Finally, the expressions for \mathcal{I}_+ read

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: \frac{1}{2}E_{+\bar{\beta}} - 6iLF_{+\bar{\beta}\gamma}{}^{\gamma} + 8\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}BG_{+\gamma_1\cdots\gamma_3}, \\
\Gamma^+ &: \frac{1}{2}E_{++}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -iLF_{+\bar{\beta}_1\cdots\bar{\beta}_3} + \frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}{}^{\gamma_1}(LG_{+\gamma_1} + 24BG_{+\gamma_1\gamma_2}{}^{\gamma_2}), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: 0, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: 0.
\end{aligned} \tag{C.5}$$

C.2 Integrability conditions on e_{ij}

The expressions for the integrability condition \mathcal{I} on the basis element e_{ij} read

$$\begin{aligned}
1 &: -\epsilon^{b_1b_2}(LG_{b_1b_2} + 24BG_{b_1b_2r}{}^r + 24BG_{b_1b_2-+}) + 2\epsilon^{r_1r_2}BP_{r_1r_2}, \\
\Gamma^{\bar{b}_1\bar{b}_2} &: \frac{1}{4}\epsilon_{\bar{b}_1\bar{b}_2}(-LG_c{}^c + LG_r{}^r + LG_{-+} + 12BG_{c_1}{}^{c_1}c_2{}^{c_2} - 24BG_c{}^c r{}^r - 24BG_c{}^c - + + \\
&\quad + 12BG_{r_1}{}^{r_1}r_2{}^{r_2} + 24BG_r{}^r - +), \\
\Gamma^{\bar{b}\bar{q}} &: -\epsilon_{\bar{b}}^{c_1}(LG_{c_1\bar{q}} - 24BG_{c_1c_2}{}^{c_2}\bar{q} + 24BG_{c_1\bar{q}r}{}^r + 24BG_{c_1\bar{q}-+}) + 2\epsilon_{\bar{q}}^r BP_{\bar{b}r}, \\
\Gamma^{\bar{b}+} &: -\epsilon_{\bar{b}}^{c_1}(LG_{c_1+} - 24BG_{c_1c_2}{}^{c_2} + 24BG_{c_1+r}{}^r), \\
\Gamma^{\bar{q}_1\bar{q}_2} &: -12\epsilon^{b_1b_2}BG_{b_1b_2\bar{q}_1\bar{q}_2} - \frac{1}{4}\epsilon_{\bar{q}_1\bar{q}_2}(LP + 2BP_c{}^c - 2BP_r{}^r + 2BP_{-+}), \\
\Gamma^{\bar{q}+} &: -24\epsilon^{b_1b_2}BG_{b_1b_2\bar{q}+} + 2\epsilon_{\bar{q}}^r BP_{r+}, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}_1\bar{q}_1} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2}(LG_{\bar{q}_1\bar{q}_2} - 24BG_{\bar{q}_1\bar{q}_2c}{}^c + 24BG_{\bar{q}_1\bar{q}_2-+}) - \frac{1}{4}\epsilon_{\bar{q}_1\bar{q}_2}BP_{\bar{b}_1\bar{b}_2}, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}+} &: \frac{1}{4}\epsilon_{\bar{b}_1\bar{b}_2}(LG_{\bar{q}+} - 24BG_{\bar{q}+c}{}^c + 24BG_{\bar{q}+r}{}^r), \\
\Gamma^{\bar{b}\bar{q}_1\bar{q}_2+} &: -12\epsilon_{\bar{b}}^c BG_{c\bar{q}_1\bar{q}_2+} - \frac{1}{2}\epsilon_{\bar{q}_1\bar{q}_2}BP_{\bar{b}+}.
\end{aligned} \tag{C.6}$$

The integrability conditions \mathcal{I}_a are given by

$$\begin{aligned}
\Gamma^{\bar{b}} &: -\frac{1}{2}\epsilon_{\bar{b}}^c(E_{ac} - 12iLF_{acr}{}^r - 12iLF_{ac-+}) - 24\epsilon^{r_1r_2}BG_{a\bar{b}r_1r_2} + \\
&\quad + 24g_{a\bar{b}}\epsilon^{r_1r_2}(BG_{r_1r_2c}{}^c + BG_{r_1r_2-+}), \\
\Gamma^{\bar{q}} &: -\epsilon_{\bar{q}}^{r_1}(LG_{ar_1} + 24BG_{ar_1c}{}^c - 24BG_{ar_1r_2}{}^{r_2} + 24BG_{ar_1-+}), \\
\Gamma^+ &: -24\epsilon^{r_1r_2}BG_{ar_1r_2+}, \\
\Gamma^{\bar{b}_1\bar{b}_2\bar{q}} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2}(E_{a\bar{q}} + 12iLF_{a\bar{q}c}{}^c - 12iLF_{a\bar{q}r}{}^r - 12iLF_{a\bar{q}-+}) + \\
&\quad + 24\epsilon_{\bar{q}}^{r_1}g_{a[\bar{b}_1}(-2BG_{\bar{b}_2]r_1c}{}^c - BG_{\bar{b}_2]r_1r_2}{}^{r_2} + BG_{\bar{b}_2]r_1-+), \\
\Gamma^{\bar{b}_1\bar{b}_2+} &: \frac{1}{8}\epsilon_{\bar{b}_1\bar{b}_2}(E_{a+} + 12iLF_{a+c}{}^c - 12iLF_{a+r}{}^r) + 24g_{a[\bar{b}_2}\epsilon^{r_1r_2}BG_{\bar{b}_2]r_1r_2+}, \\
\Gamma^{\bar{b}\bar{q}_1\bar{q}_2} &: 6\epsilon_{\bar{q}_1\bar{q}_2}(BG_{a\bar{b}c}{}^c - BG_{a\bar{b}r}{}^r + BG_{a\bar{b}-+}) - 3g_{a\bar{b}}\epsilon_{\bar{q}_1\bar{q}_2}(BG_{c_1}{}^{c_1}c_2{}^{c_2} - 2BG_c{}^c r{}^r + \\
&\quad + 2BG_c{}^c - + + BG_{r_1}{}^{r_1}r_2{}^{r_2} - 2BG_r{}^r - +), \\
\Gamma^{\bar{b}\bar{q}+} &: 6i\epsilon_{\bar{b}}^c LF_{ac\bar{q}+} + 24g_{a\bar{b}}\epsilon_{\bar{q}}^{r_1}(BG_c{}^c r_1+ - BG_{r_1r_2}{}^{r_2} +) - 24\epsilon_{\bar{q}}^r BG_{a\bar{b}r+}, \\
\Gamma^{\bar{q}_1\bar{q}_2+} &: \frac{1}{4}\epsilon_{\bar{q}_1\bar{q}_2}(LG_{a+} + 24BG_{a+c}{}^c - 24BG_{a+r}{}^r),
\end{aligned}$$

$$\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} : -\frac{3}{4} i \epsilon_{\bar{b}_1 \bar{b}_2} L F_{a \bar{q}_1 \bar{q}_2 +} + 24 \epsilon_{\bar{q}_1 \bar{q}_2} g_{a[\bar{b}_1} B G_{\bar{b}_2] +}{}^r. \quad (C.7)$$

Similarly, $\mathcal{I}_{\bar{a}}$ reads

$$\begin{aligned} \Gamma^{\bar{b}} &: -\frac{1}{2} \epsilon_{\bar{b}}^{c_1} (E_{\bar{a} c_1} + 12i L F_{\bar{a} c_1 c_2}{}^{c_2} - 12i L F_{\bar{a} c_1}{}^r - 12i L F_{\bar{a} c_1 - +}) + 24 \epsilon^{r_1 r_2} B G_{\bar{a} \bar{b} r_1 r_2}, \\ \Gamma^{\bar{q}} &: 6i \epsilon^{b_1 b_2} L F_{\bar{a} b_1 b_2 \bar{q}} - \epsilon_{\bar{q}}^{r_1} (L G_{\bar{a} r_1} - 24 B G_{\bar{a} c}{}^c{}_{r_1} + 24 B G_{\bar{a} r_1 r_2}{}^{r_2} - 24 B G_{\bar{a} r_1 - +}), \\ \Gamma^+ &: 6i \epsilon^{b_1 b_2} L F_{\bar{a} b_1 b_2 +} + 24 \epsilon^{r_1 r_2} B G_{\bar{a} r_1 r_2 +}, \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{a} \bar{q}} + 12i L F_{\bar{a} \bar{q} c}{}^c - 12i L F_{\bar{a} \bar{q} r}{}^r - 12i L F_{\bar{a} \bar{q} - +}), \\ \Gamma^{\bar{b}_1 \bar{b}_2 +} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{a} +} + 12i L F_{\bar{a} + c}{}^c - 12i L F_{\bar{a} + r}{}^r), \\ \Gamma^{\bar{b} \bar{q}_1 \bar{q}_2} &: 3i \epsilon_{\bar{b}}^c L F_{\bar{a} c \bar{q}_1 \bar{q}_2} + \frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{\bar{a} \bar{b}} + 24 B G_{\bar{a} \bar{b} r}{}^r - 24 B G_{\bar{a} \bar{b} - +}), \\ \Gamma^{\bar{b} \bar{q} +} &: 6i \epsilon_{\bar{b}}^c L F_{\bar{a} c \bar{q} +} + 24 \epsilon_{\bar{q}}^r B G_{\bar{a} \bar{b} r +}, \\ \Gamma^{\bar{q}_1 \bar{q}_2 +} &: \frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{\bar{a} +} - 24 B G_{\bar{a} + c}{}^c + 24 B G_{\bar{a} + r}{}^r), \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} &: -\frac{3}{4} i \epsilon_{\bar{b}_1 \bar{b}_2} L F_{\bar{a} \bar{q}_1 \bar{q}_2 +}. \end{aligned} \quad (C.8)$$

The integrability condition \mathcal{I}_M with $M = p$ is given by

$$\begin{aligned} \Gamma^{\bar{b}} &: -\frac{1}{2} \epsilon_{\bar{b}}^{c_1} (E_{c_1 p} - 12i L F_{c_1 c_2}{}^{c_2}{}_p + 12i L F_{c_1 p r}{}^r + 12i L F_{c_1 p - +}), \\ \Gamma^{\bar{q}} &: 6i \epsilon^{b_1 b_2} L F_{b_1 b_2 p \bar{q}} - \epsilon_{\bar{q}}^r (L G_{p r} - 24 B G_{p r c}{}^c - 24 B G_{p r - +}), \\ \Gamma^+ &: 6i \epsilon^{b_1 b_2} L F_{b_1 b_2 p +}, \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{p \bar{q}} + 12i L F_{p \bar{q} c}{}^c - 12i L F_{p \bar{q} r}{}^r - 12i L F_{p \bar{q} - +}) + 12 \epsilon_{\bar{q}}^r B G_{\bar{b}_1 \bar{b}_2 p r}, \\ \Gamma^{\bar{b}_1 \bar{b}_2 +} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{p +} + 12i L F_{p + c}{}^c - 12i L F_{p + r}{}^r), \\ \Gamma^{\bar{b} \bar{q}_1 \bar{q}_2} &: -3i \epsilon_{\bar{b}}^c L F_{c p \bar{q}_1 \bar{q}_2} - \frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{\bar{b} p} - 24 B G_{\bar{b} p c}{}^c + 24 B G_{\bar{b} p r}{}^r - 24 B G_{\bar{b} p - +}), \\ \Gamma^{\bar{b} \bar{q} +} &: -6i \epsilon_{\bar{b}}^c L F_{c p \bar{q} +} - 24 \epsilon_{\bar{q}}^r B G_{\bar{b} p r +}, \\ \Gamma^{\bar{q}_1 \bar{q}_2 +} &: \frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{p +} - 24 B G_{p + c}{}^c + 24 B G_{p + r}{}^r), \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} &: -\frac{3}{4} i \epsilon_{\bar{b}_1 \bar{b}_2} L F_{p \bar{q}_1 \bar{q}_2 +} - 3 \epsilon_{\bar{q}_1 \bar{q}_2} B G_{\bar{b}_1 \bar{b}_2 p +}. \end{aligned} \quad (C.9)$$

Furthermore, \mathcal{I}_M with $M = \bar{p}$ is given by the expressions

$$\begin{aligned} \Gamma^{\bar{b}} &: -\frac{1}{2} \epsilon_{\bar{b}}^{c_1} (E_{c_1 \bar{p}} - 12i L F_{c_1 c_2}{}^{c_2}{}_{\bar{p}} + 12i L F_{c_1 \bar{p} r}{}^r + 12i L F_{c_1 \bar{p} - +}) + \\ &\quad -48 \epsilon_{\bar{p}}^{r_1} (B G_{\bar{b} r_1 c}{}^c + 2 B G_{\bar{b} r_1 r_2}{}^{r_2} + B G_{\bar{c} r_1 - +}), \\ \Gamma^{\bar{q}} &: -24 \epsilon_{\bar{q}}^{r_1} (B G_{\bar{p} r_1 c}{}^c - B G_{\bar{p} r_1 r_2}{}^{r_2} + B G_{\bar{p} r_1 - +}) + \\ &\quad -12 \epsilon_{\bar{p} \bar{q}} (B G_{c_1}{}^{c_1}{}_{c_2}{}^{c_2} - 2 B G_c{}^c{}_{r}{}^r + 2 B G_c{}^c{}_{- +} + B G_{r_1}{}^{r_1}{}_{r_2}{}^{r_2} - 2 B G_r{}^r{}_{- +}), \\ \Gamma^+ &: 6i \epsilon^{b_1 b_2} L F_{b_1 b_2 \bar{p} +} + 48 \epsilon_{\bar{p}}^r B G_{r + c}{}^c, \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{p} \bar{q}} + 12i L F_{\bar{p} \bar{q} c}{}^c - 12i L F_{\bar{p} \bar{q} - +}) - 12 \epsilon_{\bar{q}}^r B G_{\bar{b}_1 \bar{b}_2 \bar{p} r} + \\ &\quad + 12 \epsilon_{\bar{p} \bar{q}} (B G_{\bar{b}_1 \bar{b}_2 r}{}^r - B G_{\bar{b}_1 \bar{b}_2 - +}), \\ \Gamma^{\bar{b}_1 \bar{b}_2 +} &: \frac{1}{8} \epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{p} +} + 12i L F_{\bar{p} + c}{}^c - 12i L F_{\bar{p} + r}{}^r) + 24 \epsilon_{\bar{p}}^r B G_{\bar{b}_1 \bar{b}_2 r +}, \\ \Gamma^{\bar{b} \bar{q}_1 \bar{q}_2} &: -\frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{\bar{b} \bar{p}} + 24 B G_{\bar{b} \bar{p} c}{}^c - 24 B G_{\bar{b} \bar{p} r}{}^r + 24 B G_{\bar{b} \bar{p} - +}), \\ \Gamma^{\bar{b} \bar{q} +} &: -6i \epsilon_{\bar{b}}^c L F_{c \bar{p} \bar{q} +} + 24 \epsilon_{\bar{p} \bar{q}} (B G_{\bar{b} + c}{}^c - B G_{\bar{b} + r}{}^r) + 24 \epsilon_{\bar{q}}^r B G_{\bar{b} \bar{p} r +}, \\ \Gamma^{\bar{q}_1 \bar{q}_2 +} &: \frac{1}{4} \epsilon_{\bar{q}_1 \bar{q}_2} (L G_{\bar{p} +} + 24 B G_{\bar{p} + c}{}^c - 24 B G_{\bar{p} + r}{}^r), \\ \Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} &: 3 \epsilon_{\bar{q}_1 \bar{q}_2} B G_{\bar{b}_1 \bar{b}_2 \bar{p} +}. \end{aligned} \quad (C.10)$$

The integrability conditions \mathcal{I}_- read

$$\begin{aligned} \Gamma^{\bar{b}} &: -\frac{1}{2} \epsilon_{\bar{b}}^{c_1} (E_{c_1 -} - 12i L F_{c_1 c_2}{}^{c_2}{}_{-} + 12i L F_{c_1 - r}{}^r) + 24 \epsilon^{r_1 r_2} B G_{\bar{b} r_1 r_2 -}, \\ \Gamma^{\bar{q}} &: -6i \epsilon^{b_1 b_2} L F_{b_1 b_2 \bar{q} -} + \epsilon_{\bar{q}}^r L G_{r -} + 24 \epsilon_{\bar{q}}^{r_1} (B G_c{}^c{}_{r_1 -} - B G_{r_1 r_2}{}^{r_2}{}_{-}), \end{aligned}$$

$$\begin{aligned}
\Gamma^+ &: 6i\epsilon^{b_1 b_2} LF_{b_1 b_2 - +} + 24\epsilon^{r_1 r_2} BG_{r_1 r_2 c}^c, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: \frac{1}{8}\epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{q}-} - 12iLF_{\bar{q}-c}^c + 12iLF_{\bar{q}-r}^r) + 12\epsilon_{\bar{q}}^r BG_{\bar{b}_1 \bar{b}_2 r -}, \\
\Gamma^{\bar{b}_1 \bar{b}_2 +} &: \frac{1}{8}\epsilon_{\bar{b}_1 \bar{b}_2} (E_{-+} + 12iLF_{c-+}^c - 12iLF_{r-+}^r) + 12\epsilon^{r_1 r_2} BG_{\bar{b}_1 \bar{b}_2 r_1 r_2}, \\
\Gamma^{\bar{q}_1 \bar{q}_2} &: -3i\epsilon_{\bar{b}}^c LG_{c\bar{q}_1 \bar{q}_2 -} - \frac{1}{4}\epsilon_{\bar{q}_1 \bar{q}_2} (LG_{\bar{b}-} + 24BG_{\bar{b}-c}^c - 24BG_{\bar{b}-r}^r), \\
\Gamma^{\bar{b} \bar{q} +} &: 6i\epsilon_{\bar{b}}^c LF_{c\bar{q}-+} + 12\epsilon_{\bar{q}}^{r_1} (BG_{\bar{b} r_1 c}^c - BG_{\bar{b} r_1 r_2}^{r_2}), \\
\Gamma^{\bar{q}_1 \bar{q}_2 +} &: \frac{1}{4}\epsilon_{\bar{q}_1 \bar{q}_2} (LG_{-+} - 12BG_{c_1 c_2}^{c_1 c_2} + 24BG_{c r}^c - 12BG_{r_1 r_2}^{r_1 r_2}), \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} &: -\frac{3}{4}i\epsilon_{\bar{b}_1 \bar{b}_2} LF_{\bar{q}_1 \bar{q}_2 - +} + 3\epsilon_{\bar{q}_1 \bar{q}_2} BG_{\bar{b}_1 \bar{b}_2 r}^r.
\end{aligned} \tag{C.11}$$

Finally, for \mathcal{I}_+ we find

$$\begin{aligned}
\Gamma^{\bar{b}} &: -\frac{1}{2}\epsilon_{\bar{b}}^{c_1} (E_{c_1 +} - 12iLF_{c_1 c_2}^{c_2} + 12iLF_{c_1 + r}^r) - 24\epsilon^{r_1 r_2} BG_{\bar{b} r_1 r_2 +}, \\
\Gamma^{\bar{q}} &: -6i\epsilon^{b_1 b_2} LF_{b_1 b_2 \bar{q} +} + \epsilon_{\bar{q}}^{r_1} (LG_{r_1 +} - 24BG_{r_1 + c}^c + 24BG_{r_1 r_2}^{r_2 +}), \\
\Gamma^+ &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}} &: \frac{1}{8}\epsilon_{\bar{b}_1 \bar{b}_2} (E_{\bar{q} +} - 12iLF_{\bar{q} + c}^c + 12iLF_{\bar{q} + r}^r) - 12\epsilon_{\bar{q}}^r BG_{\bar{b}_1 \bar{b}_2 r +}, \\
\Gamma^{\bar{b}_1 \bar{b}_2 +} &: \frac{1}{8}\epsilon_{\bar{b}_1 \bar{b}_2} E_{++}, \\
\Gamma^{\bar{b} \bar{q} +} &: 0, \\
\Gamma^{\bar{q}_1 \bar{q}_2} &: -3i\epsilon_{\bar{b}}^c LF_{c\bar{q}_1 \bar{q}_2 +} - \frac{1}{4}\epsilon_{\bar{q}_1 \bar{q}_2} (LG_{\bar{b} +} - 24BG_{\bar{b} + c}^c + 24BG_{\bar{b} + r}^r), \\
\Gamma^{\bar{q}_1 \bar{q}_2 +} &: 0, \\
\Gamma^{\bar{b}_1 \bar{b}_2 \bar{q}_1 \bar{q}_2 +} &: 0.
\end{aligned} \tag{C.12}$$

C.3 Integrability conditions on e_{k5}

The expressions for the integrability condition \mathcal{I} on the basis element e_{k5} read

$$\begin{aligned}
1 &: -2LG_{k-} - 48LG_{k-\sigma}^\sigma, \\
\Gamma^{\bar{k}\bar{\tau}} &: LG_{\bar{\tau}-} - 24BG_{k\bar{k}\bar{\tau}-} + 24BG_{\bar{\tau}-\sigma}^\sigma, \\
\Gamma^{\bar{k}+} &: -\frac{1}{2}(LG_{k\bar{k}} - LG_{\sigma}^\sigma + LG_{-+}) - 12BG_{k\bar{k}\sigma}^\sigma + 12BG_{k\bar{k}-+} + 6BG_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2} - 12BG_{\sigma}^\sigma - +, \\
\Gamma^{\bar{\tau}_1 \bar{\tau}_2} &: -24BG_{k\bar{\tau}_1 \bar{\tau}_2 -} - \epsilon_{\bar{\tau}_1 \bar{\tau}_2}^\sigma BP_{\sigma-}, \\
\Gamma^{\bar{\tau}+} &: -LG_{k\bar{\tau}} - 24BG_{k\bar{\tau}\sigma}^\sigma + 24BG_{k\bar{\tau}-+} - \epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} BP_{\sigma_1 \sigma_2}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3} &: 4BG_{\bar{\tau}_1 \dots \bar{\tau}_3 -} - \frac{1}{6}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} BP_{\bar{k}-}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \bar{\tau}_2 +} &: \frac{1}{4}LG_{\bar{\tau}_1 \bar{\tau}_2} - 6BG_{k\bar{k}\bar{\tau}_1 \bar{\tau}_2} + 6BG_{\bar{\tau}_1 \bar{\tau}_2 \sigma}^\sigma - 6BG_{\bar{\tau}_1 \bar{\tau}_2 -+} + \frac{1}{2}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^\sigma BP_{\bar{k}\sigma}, \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3 +} &: -4BG_{k\bar{\tau}_1 \dots \bar{\tau}_3} + \frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (LP + 2BP_{k\bar{k}} - 2BP_{\sigma}^\sigma - 2BP_{-+}).
\end{aligned} \tag{C.13}$$

The integrability conditions \mathcal{I}_k are given by

$$\begin{aligned}
\Gamma^{\bar{k}} &: -\frac{1}{2}E_{k-} + 6iLF_{k-\sigma}^\sigma + 16\epsilon^{\sigma_1 \dots \sigma_3} BG_{\sigma_1 \dots \sigma_3 -}, \\
\Gamma^{\bar{\tau}} &: -24\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} BG_{k\sigma_1 \sigma_2 -}, \\
\Gamma^+ &: \frac{1}{2}E_{kk} + 8\epsilon^{\sigma_1 \dots \sigma_3} BG_{k\sigma_1 \dots \sigma_3}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \bar{\tau}_2} &: 3iLF_{k\bar{\tau}_1 \bar{\tau}_2 -} + 12\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} BG_{\sigma_1 \sigma_2}^{\sigma_2} - , \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{k\bar{\tau}} + 3iLF_{k\bar{\tau}\sigma}^\sigma - 3iLF_{k\bar{\tau}-+} + 12\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} (BG_{\sigma_1 \sigma_2 \sigma_3}^{\sigma_3} + BG_{\sigma_1 \sigma_2 -+}), \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3} &: \frac{1}{12}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (LG_{k-} - 24BG_{k-\sigma}^\sigma), \\
\Gamma^{\bar{\tau}_1 \bar{\tau}_2 +} &: -\frac{1}{4}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} (LG_{k\sigma_1} - 24BG_{k\sigma_1 \sigma_2}^{\sigma_2} - 24BG_{k\sigma_1 -+}), \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3 +} &: \frac{1}{2}iLF_{k\bar{\tau}_1 \dots \bar{\tau}_3} - \frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (LG_{k\bar{k}} - 12BG_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2} - 24BG_{\sigma}^\sigma - +).
\end{aligned} \tag{C.14}$$

Similarly, $\mathcal{I}_{\bar{k}}$ reads

$$\Gamma^{\bar{k}} : -\frac{1}{2}E_{\bar{k}-} + 6iLF_{\bar{k}-\sigma}^\sigma,$$

$$\begin{aligned}
\Gamma^{\bar{\tau}} &: 12iLF_{k\bar{k}\bar{\tau}-} + 24\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{k\sigma_1\sigma_2-}, \\
\Gamma^+ &: \frac{1}{2}E_{k\bar{k}} + 6iLF_{k\bar{k}\sigma}^{\sigma} - 6iLF_{k\bar{k}-+} - 8\epsilon^{\sigma_1\cdots\sigma_3}BG_{k\sigma_1\cdots\sigma_3}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: +3iLF_{\bar{k}\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{\bar{k}\bar{\tau}} + 3iLF_{\bar{k}\bar{\tau}\sigma}^{\sigma} - 3iLF_{\bar{k}\bar{\tau}-+}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{1}{12}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\bar{k}-} + 24BG_{\bar{k}-\sigma}^{\sigma}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: 3iLF_{k\bar{k}\bar{\tau}_1\bar{\tau}_2} - \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}(LG_{k\sigma} + 24BG_{k\sigma\sigma_2}^{\sigma_2} + 24BG_{k\sigma-+}), \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{2}iLF_{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3}.
\end{aligned} \tag{C.15}$$

The integrability condition \mathcal{I}_M with $M = \rho$ is given by

$$\begin{aligned}
\Gamma^{\bar{k}} &: -\frac{1}{2}E_{\rho-} - 6iLF_{k\bar{k}\rho-} + 6iLF_{\rho-\sigma}^{\sigma}, \\
\Gamma^{\bar{\tau}} &: 12iLF_{k\rho\bar{\tau}-} + 24\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{\rho\sigma_1\sigma_2-}, \\
\Gamma^+ &: \frac{1}{2}E_{\rho k} + 6iLF_{k\rho\sigma}^{\sigma} - 6iLF_{k\rho-+}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: 3iLF_{\rho\bar{\tau}_1\bar{\tau}_2-} + 12\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}BG_{\bar{k}\rho\sigma-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{\rho\bar{\tau}} - 3iLF_{\rho\bar{\tau}k\bar{k}} + 3iLF_{\rho\bar{\tau}\sigma}^{\sigma} - 3iLF_{\rho\bar{\tau}-+} + 12\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{\bar{k}\rho\sigma_1\sigma_2}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{1}{12}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\rho-} + 24BG_{\rho\sigma}^{\sigma} - 24BG_{k\bar{k}\rho-}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: 3iLF_{k\rho\bar{\tau}_1\bar{\tau}_2} - \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(LG_{\rho\sigma_1} - 24BG_{\rho\sigma_1 k\bar{k}} + 24BG_{\rho\sigma_1\sigma_2}^{\sigma_2} + 24BG_{\rho\sigma_1-+}), \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{2}iLF_{\rho\bar{\tau}_1\cdots\bar{\tau}_3} + \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{k\rho} + 24BG_{k\rho\sigma}^{\sigma} + 24BG_{k\rho-+}).
\end{aligned} \tag{C.16}$$

Furthermore, \mathcal{I}_M with $M = \bar{\rho}$ is given by the expressions

$$\begin{aligned}
\Gamma^{\bar{k}} &: -\frac{1}{2}E_{\bar{\rho}-} + 6iLF_{\bar{\rho}-\sigma}^{\sigma} - 6iLF_{k\bar{k}\bar{\rho}-} - 48\epsilon_{\bar{\rho}}^{\sigma_1\sigma_2}BG_{k\sigma_1\sigma_2-}, \\
\Gamma^{\bar{\tau}} &: 12iLF_{k\bar{\rho}\bar{\tau}-} - 48\epsilon_{\bar{\rho}\bar{\tau}}^{\sigma_1}(BG_{k\bar{k}\sigma_1-} - BG_{\sigma_1\sigma_2}^{\sigma_2-}) - 24\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{\bar{\rho}\sigma_1\sigma_2-}, \\
\Gamma^+ &: \frac{1}{2}E_{k\bar{\rho}} + 6iLF_{k\bar{\rho}\sigma}^{\sigma} - 6iLF_{k\bar{\rho}-+} - 24\epsilon_{\bar{\rho}}^{\sigma_1\sigma_2}(BG_{k\bar{k}\sigma_1\sigma_2} - BG_{\sigma_1\sigma_2-+}), \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: 3iLF_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2-} - 12\epsilon_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2}^{\sigma}BG_{\bar{k}-\sigma}^{\sigma} - 12\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}BG_{\bar{k}\bar{\rho}\sigma-+}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{\bar{\rho}\bar{\tau}} - 3iLF_{k\bar{k}\bar{\rho}\bar{\tau}} + 3iLF_{\bar{\rho}\bar{\tau}\sigma}^{\sigma} - 3iLF_{\bar{\rho}\bar{\tau}-+} - 12\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{\bar{k}\bar{\rho}\sigma_1\sigma_2} + \\
&\quad + 24\epsilon_{\bar{\rho}\bar{\tau}}^{\sigma_1}(BG_{\bar{k}\sigma_1\sigma_2}^{\sigma_2} + BG_{\bar{k}\sigma_1-+}), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: \frac{1}{12}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\bar{\rho}-} - 24BG_{\bar{\rho}-\sigma}^{\sigma} + 24BG_{k\bar{k}\bar{\rho}-}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: 3iLF_{k\bar{\rho}\bar{\tau}_1\bar{\tau}_2} - \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma_1}(LG_{\bar{\rho}\sigma_1} + 24BG_{k\bar{k}\bar{\rho}\sigma_1} - 24BG_{\bar{\rho}\sigma_1\sigma_2}^{\sigma_2} - 24BG_{\bar{\rho}\sigma_1-+}) + \\
&\quad + 3\epsilon_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2}(-2BG_{k\bar{k}\sigma}^{\sigma} - 2BG_{k\bar{k}-+} + BG_{\sigma_1}^{\sigma_1\sigma_2} + 2BG_{\sigma}^{\sigma-+}), \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{2}iLF_{\bar{\rho}\bar{\tau}_1\cdots\bar{\tau}_3} + \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\bar{k}\bar{\rho}} - 24BG_{\bar{k}\bar{\rho}\sigma}^{\sigma} - 24BG_{\bar{k}\bar{\rho}-+}).
\end{aligned} \tag{C.17}$$

The integrability conditions \mathcal{I}_- read

$$\begin{aligned}
\Gamma^{\bar{k}} &: -\frac{1}{2}E_{--}, \\
\Gamma^{\bar{\tau}} &: 0, \\
\Gamma^+ &: \frac{1}{2}E_{k-} + 6iLF_{k-\sigma}^{\sigma} + 8\epsilon^{\sigma_1\cdots\sigma_3}BG_{\sigma_1\cdots\sigma_3-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{\bar{\tau}-} + 3iLF_{k\bar{k}\bar{\tau}-} - 3iLF_{\bar{\tau}-\sigma}^{\sigma} + 12\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{k\sigma_1\sigma_2-}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: 0, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: 3iLF_{k\bar{\tau}_1\bar{\tau}_2-} + \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}(LG_{\sigma-} - 24BG_{k\bar{k}\sigma-} + 24\sigma_{\sigma-\sigma_2}^{\sigma_2}), \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: -\frac{1}{2}iLF_{\bar{\tau}_1\cdots\bar{\tau}_3-} + \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\bar{k}-} + 24BG_{\bar{k}-\sigma}^{\sigma}).
\end{aligned} \tag{C.18}$$

Finally, for \mathcal{I}_+ we find

$$\Gamma^{\bar{k}} : -\frac{1}{2}E_{-+} + 6iLF_{k\bar{k}-+} - 6iLF_{\sigma-+}^{\sigma} + 16\epsilon^{\sigma_1\cdots\sigma_3}BG_{k\sigma_1\cdots\sigma_3},$$

$$\begin{aligned}
\Gamma^{\bar{\tau}} &: 12iLF_{k\bar{\tau}-+} + 24\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}(BG_{k\bar{k}\sigma_1\sigma_2} - BG_{\sigma_1\sigma_2\sigma_3}^{\sigma_3}), \\
\Gamma^+ &: \frac{1}{2}E_{k+} + 6iLF_{k+\sigma}^{\sigma} - 8\epsilon^{\sigma_1\cdots\sigma_3}BG_{\sigma_1\cdots\sigma_3+}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -3iLF_{\bar{\tau}_1\bar{\tau}_2-+} + 12\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}BG_{k\sigma\sigma_2}^{\sigma_2}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -\frac{1}{4}E_{\bar{\tau}+} + 3iLF_{k\bar{k}\bar{\tau}+} - 3iLF_{\bar{\tau}+\sigma}^{\sigma} - 12\epsilon_{\bar{\tau}}^{\sigma_1\sigma_2}BG_{k\sigma_1\sigma_2+}, \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{12}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{-+} - 24BG_{k\bar{k}\sigma}^{\sigma} + 12BG_{\sigma_1}^{\sigma_1\sigma_2\sigma_2}), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: 3iLF_{k\bar{\tau}_1\bar{\tau}_2+} + \frac{1}{4}\epsilon_{\bar{\tau}_1\bar{\tau}_2}^{\sigma}(LG_{\sigma+} + 24BG_{k\bar{k}\sigma+} - 24BG_{\sigma+\sigma_2}^{\sigma_2}), \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: -\frac{1}{2}iLF_{\bar{\tau}_1\cdots\bar{\tau}_3+} + \frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(LG_{\bar{k}+} - 24BG_{\bar{k}+\sigma}^{\sigma}). \tag{C.19}
\end{aligned}$$

C.4 Integrability conditions on e_{1234}

The expressions for the integrability condition \mathcal{I} on the basis element e_{1234} read

$$\begin{aligned}
1 &: 4\epsilon^{\gamma_1\cdots\gamma_4}BG_{\gamma_1\cdots\gamma_4} + LP + 2BP_{\gamma}^{\gamma} + 2BP_{-+}, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2} &: -\frac{1}{4}\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}(LG_{\gamma_1\gamma_2} - 24BG_{\gamma_1\gamma_2\gamma_3}^{\gamma_3} + 24BG_{\gamma_1\gamma_2-+}) + BP_{\bar{\beta}_1\bar{\beta}_2}, \\
\Gamma^{\bar{\beta}+} &: 8\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}BG_{\gamma_1\cdots\gamma_3+} + 2BP_{\bar{\beta}+}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(-LG_{\gamma}^{\gamma} + LG_{-+} + 12BG_{\gamma_1}^{\gamma_1\gamma_2\gamma_2} - 24BG_{\gamma}^{\gamma-+}), \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3+} &: -\frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^{\gamma_1}(LG_{\gamma_1+} - 24BG_{\gamma_1+\gamma_2}^{\gamma_2}). \tag{C.20}
\end{aligned}$$

The integrability conditions \mathcal{I}_{α} are

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: -2i\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{\alpha\gamma_1\cdots\gamma_3} - LG_{\alpha\bar{\beta}} - 24(BG_{\alpha\bar{\beta}\gamma}^{\gamma} + BG_{\alpha\bar{\beta}-+}) + \\
&\quad + 12g_{\alpha\bar{\beta}}(BG_{\gamma_1}^{\gamma_1\gamma_2\gamma_2} + 2BG_{\gamma}^{\gamma-+}), \\
\Gamma^+ &: -LG_{\alpha+} - 24BG_{\alpha+\gamma}^{\gamma}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -\frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^{\gamma_1}(\frac{1}{2}E_{\alpha\gamma_1} + 6iLF_{\alpha\gamma_1\gamma_2}^{\gamma_2} - 6iLF_{\alpha\gamma_1-+}) - 4BG_{\alpha\bar{\beta}_1\cdots\bar{\beta}_3} + \\
&\quad + 12g_{\alpha[\bar{\beta}_1}(BG_{\bar{\beta}_2\bar{\beta}_3]\gamma}^{\gamma} + BG_{\bar{\beta}_2\bar{\beta}_3]-+), \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: \frac{3}{2}i\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}LF_{\alpha\gamma_1\gamma_2+} - 12BG_{\alpha\bar{\beta}_1\bar{\beta}_2+} + 24g_{\alpha[\bar{\beta}_1}BG_{\bar{\beta}_2]\gamma}^{\gamma}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(\frac{1}{2}E_{\alpha+} + 6iLF_{\alpha+\gamma}^{\gamma}) + 4g_{\alpha[\bar{\beta}_1}BG_{\bar{\beta}_2\cdots\bar{\beta}_4]\gamma}^{\gamma}. \tag{C.21}
\end{aligned}$$

The integrability conditions $\mathcal{I}_{\bar{\alpha}}$ read

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: -2i\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{\bar{\alpha}\gamma_1\cdots\gamma_3} - LG_{\bar{\alpha}\bar{\beta}} + 24(BG_{\bar{\alpha}\bar{\beta}\gamma}^{\gamma} + BG_{\bar{\alpha}\bar{\beta}-+}), \\
\Gamma^+ &: -LG_{\bar{\alpha}+} + 24BG_{\bar{\alpha}+\gamma}^{\gamma}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -\frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^{\gamma_1}(\frac{1}{2}E_{\bar{\alpha}\gamma_1} + 6i(LF_{\bar{\alpha}\gamma_1\gamma_2}^{\gamma_2} - LF_{\bar{\alpha}\gamma_1-+})) + 4BG_{\bar{\alpha}\bar{\beta}_1\cdots\bar{\beta}_3}, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: \frac{3}{2}i\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}LF_{\bar{\alpha}\gamma_1\gamma_2+} + 12BG_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2+}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(\frac{1}{2}E_{\bar{\alpha}+} + 6iLF_{\bar{\alpha}+\gamma}^{\gamma}). \tag{C.22}
\end{aligned}$$

Similarly, \mathcal{I}_A with $A = -$ is given by

$$\begin{aligned}
\Gamma^{\bar{\beta}} &: -2i\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{-\gamma_1\cdots\gamma_3} + LG_{\bar{\beta}-} + 24BG_{\bar{\beta}-\gamma}^{\gamma}, \\
\Gamma^+ &: -LG_{-+} + 12BG_{\gamma_1}^{\gamma_1\gamma_2\gamma_2}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_3} &: -\frac{1}{12}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_3}^{\gamma_1}(\frac{1}{2}E_{\gamma_1-} - 6iLF_{\gamma_1-\gamma_2}^{\gamma_2}) + 4BG_{\bar{\beta}_1\cdots\bar{\beta}_3-}, \\
\Gamma^{\bar{\beta}_1\bar{\beta}_2+} &: \frac{3}{2}i\epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2}LF_{\gamma_1\gamma_2-+} + 12BG_{\bar{\beta}_1\bar{\beta}_2\gamma}^{\gamma}, \\
\Gamma^{\bar{\beta}_1\cdots\bar{\beta}_4+} &: \frac{1}{96}\epsilon_{\bar{\beta}_1\cdots\bar{\beta}_4}(\frac{1}{2}E_{-+} + 6iLF_{-+\gamma}^{\gamma}) + BG_{\bar{\beta}_1\cdots\bar{\beta}_4}. \tag{C.23}
\end{aligned}$$

Finally, the expressions for \mathcal{I}_+ read

$$\Gamma^{\bar{\beta}} : 2i\epsilon_{\bar{\beta}}^{\gamma_1\cdots\gamma_3}LF_{\gamma_1\cdots\gamma_3+} + LG_{\bar{\beta}+} - 24BG_{\bar{\beta}+\gamma}^{\gamma},$$

$$\begin{aligned}
\Gamma^+ &: 0, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_3} &: -\frac{1}{12} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_3}^{\gamma_1} \left(\frac{1}{2} E_{\gamma_1 +} - 6iLF_{\gamma_1 + \gamma_2}^{\gamma_2} \right) - 4BG_{\bar{\beta}_1 \dots \bar{\beta}_3 +}, \\
\Gamma^{\bar{\beta}_1 \bar{\beta}_2 +} &: 0, \\
\Gamma^{\bar{\beta}_1 \dots \bar{\beta}_4 +} &: \frac{1}{192} \epsilon_{\bar{\beta}_1 \dots \bar{\beta}_4} E_{++}.
\end{aligned} \tag{C.24}$$

C.5 Integrability conditions on $e_{i_1 \dots i_3 5}$

The expressions for the integrability condition \mathcal{I} on the basis element $e_{i_1 \dots i_3 5}$ read

$$\begin{aligned}
1 &: 16\epsilon^{\sigma_1 \dots \sigma_3} BG_{\sigma_1 \dots \sigma_3 -} - 4BP_{k-}, \\
\Gamma^{\bar{k}\bar{\tau}} &: 24\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} BG_{\bar{k}\sigma_1 \sigma_2 -} + 2BP_{\bar{\tau}-}, \\
\Gamma^{\bar{k}+} &: -8\epsilon^{\sigma_1 \dots \sigma_3} BG_{\bar{k}\sigma_1 \dots \sigma_3} + \frac{1}{2}LP - BP_{k\bar{k}} + BP_{\sigma}^{\sigma} - BP_{-+}, \\
\Gamma^{\bar{\tau}_1 \bar{\tau}_2} &: -\frac{1}{2}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} (LG_{\sigma_1 -} + 24BG_{\bar{k}\sigma_1 -} - 24BG_{\sigma_1 \sigma_2}^{\sigma_2 -}), \\
\Gamma^{\bar{\tau}+} &: -\frac{1}{2}\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} (LG_{\sigma_1 \sigma_2} + 24BG_{\bar{k}\sigma_1 \sigma_2} - 24BG_{\sigma_1 \sigma_2 \sigma_3}^{\sigma_3} - 24BG_{\sigma_1 \sigma_2 -+}) - 2BP_{k\bar{\tau}}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3} &: -\frac{1}{12}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (LG_{\bar{k}-} - 24BG_{\bar{k}-\sigma}^{\sigma}), \\
\Gamma^{\bar{k}\bar{\tau}_1 \bar{\tau}_2 +} &: \frac{1}{4}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} (LG_{\bar{k}\sigma_1} - 24BG_{\bar{k}\sigma_1 \sigma_2}^{\sigma_2} - 24BG_{\bar{k}\sigma_1 -+}) + \frac{1}{2}BP_{\bar{\tau}_1 \bar{\tau}_2}, \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3 +} &: \frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (LG_{k\bar{k}} - LG_{\sigma}^{\sigma} - LG_{-+} - 24BG_{\bar{k}\bar{k}\sigma}^{\sigma} - 24BG_{\bar{k}\bar{k}-+} + \\
&\quad + 12BG_{\sigma_1}^{\sigma_1 \sigma_2} + 24BG_{\sigma}^{\sigma -+}).
\end{aligned} \tag{C.25}$$

The integrability conditions \mathcal{I}_k are given by

$$\begin{aligned}
\Gamma^{\bar{k}} &: LG_{k-} - 24BG_{k-\sigma}^{\sigma}, \\
\Gamma^{\bar{\tau}} &: -6i\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} LF_{k\sigma_1 \sigma_2 -}, \\
\Gamma^+ &: 2i\epsilon^{\sigma_1 \dots \sigma_3} LF_{k\sigma_1 \dots \sigma_3}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \bar{\tau}_2} &: 3i\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma} LF_{\bar{k}\bar{k}\sigma -} - 12BG_{k\bar{\tau}_1 \bar{\tau}_2 -}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: 3i\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} LF_{\bar{k}\bar{k}\sigma_1 \sigma_2} + \frac{1}{2}LG_{k\bar{\tau}} - 12BG_{k\bar{\tau}\sigma}^{\sigma} + 12BG_{k\bar{\tau}-+}, \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (E_{k-} + 12iLF_{k-\sigma}^{\sigma}), \\
\Gamma^{\bar{\tau}_1 \bar{\tau}_2 +} &: \frac{1}{8}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} (E_{k\sigma_1} + 12iLF_{k\sigma_1 \sigma_2}^{\sigma_2} + 12iLF_{k\sigma_1 -+}), \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3 +} &: \frac{1}{48}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (E_{k\bar{k}} + 12iLF_{\bar{k}\bar{k}\sigma}^{\sigma} + 12iLF_{\bar{k}\bar{k}-+}) - 2BG_{k\bar{\tau}_1 \dots \bar{\tau}_3}.
\end{aligned} \tag{C.26}$$

Similarly, $\mathcal{I}_{\bar{k}}$ reads

$$\begin{aligned}
\Gamma^{\bar{k}} &: LG_{\bar{k}-} + 24BG_{\bar{k}-\sigma}^{\sigma}, \\
\Gamma^{\bar{\tau}} &: -6i\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} LF_{\bar{k}\sigma_1 \sigma_2 -} + 48BG_{\bar{\tau}-\sigma}^{\sigma}, \\
\Gamma^+ &: 2i\epsilon^{\sigma_1 \dots \sigma_3} LF_{\bar{k}\sigma_1 \dots \sigma_3} + LG_{k\bar{k}} + 12BG_{\sigma_1}^{\sigma_1 \sigma_2} - 24BG_{\sigma}^{\sigma -+}, \\
\Gamma^{\bar{k}\bar{\tau}_1 \bar{\tau}_2} &: 12BG_{\bar{k}\bar{\tau}_1 \bar{\tau}_2 -}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: \frac{1}{2}LG_{\bar{k}\bar{\tau}} + 12(BG_{\bar{k}\bar{\tau}\sigma}^{\sigma} - BG_{\bar{k}\bar{\tau}-+}), \\
\Gamma^{\bar{\tau}_1 \dots \bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} (E_{\bar{k}-} + 12iLF_{\bar{k}-\sigma}^{\sigma}) + 8BG_{\bar{\tau}_1 \dots \bar{\tau}_3 -}, \\
\Gamma^{\bar{\tau}_1 \bar{\tau}_2 +} &: \frac{1}{8}\epsilon_{\bar{\tau}_1 \bar{\tau}_2}^{\sigma_1} (E_{\bar{k}\sigma_1} + 12iLF_{\bar{k}\sigma_1 \sigma_2}^{\sigma_2} + 12iLF_{\bar{k}\sigma_1 -+}) + 12(BG_{\bar{\tau}_1 \bar{\tau}_2 \sigma}^{\sigma} - BG_{\bar{\tau}_1 \bar{\tau}_2 -+}), \\
\Gamma^{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3 +} &: \frac{1}{48}\epsilon_{\bar{\tau}_1 \dots \bar{\tau}_3} E_{\bar{k}\bar{k}} + 2BG_{\bar{k}\bar{\tau}_1 \dots \bar{\tau}_3}.
\end{aligned} \tag{C.27}$$

The integrability condition \mathcal{I}_M with $M = \rho$ is given by

$$\begin{aligned}
\Gamma^{\bar{k}} &: LG_{\rho-} - 24BG_{k\bar{k}\rho-} + 24BG_{\rho-\sigma}^{\sigma}, \\
\Gamma^{\bar{\tau}} &: -6i\epsilon_{\bar{\tau}}^{\sigma_1 \sigma_2} LF_{\rho\sigma_1 \sigma_2 -} + 48(BG_{k\rho\bar{\tau}} - g_{\rho\bar{\tau}} BG_{k-\sigma}^{\sigma}),
\end{aligned}$$

$$\begin{aligned}
\Gamma^+ &: LG_{k\rho} + 24BG_{k\rho\sigma}{}^\sigma - 24BG_{k\rho-+}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -3i\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma LF_{\bar{k}\rho\sigma-} + 24g_{\rho[\bar{\tau}_1}(BG_{\bar{\tau}_2]k\bar{k}-} - BG_{\bar{\tau}_2]-\sigma}{}^\sigma) + 12BG_{\rho\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -3i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\bar{k}\rho\sigma_1\sigma_2} + \frac{1}{2}LG_{\rho\bar{\tau}} - 12(BG_{k\bar{k}\rho\bar{\tau}} - BG_{\rho\bar{\tau}\sigma}{}^\sigma + BG_{\rho\bar{\tau}-+}) + \\
&\quad + 6g_{\rho\bar{\tau}}(2BG_{k\bar{k}\sigma}{}^\sigma - 2BG_{k\bar{k}-+} - BG_{\sigma_1}{}^{\sigma_1\sigma_2}{}_{\sigma_2} + 2BG_{\sigma}{}^\sigma{}_{-+}), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\rho-} - 12iLF_{k\bar{k}\rho-} + 12iLF_{\rho-\sigma}{}^\sigma) - 24g_{\rho[\bar{\tau}_1}BG_{\bar{\tau}_2\bar{\tau}_3]k-}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(E_{\rho\sigma_1} - 12iLF_{k\bar{k}\rho\sigma_1} + 12iLF_{\rho\sigma_1\sigma_2}{}^{\sigma_2} + 12iLF_{\rho\sigma-+}) + \\
&\quad + 24g_{\rho[\bar{\tau}_1}(BG_{\bar{\tau}_2]k\sigma}{}^\sigma - BG_{\bar{\tau}_2]k-+}) + 12BG_{k\rho\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{48}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\bar{k}\rho} - 12iLF_{\bar{k}\rho\sigma}{}^\sigma - 12iLF_{\bar{k}\rho-+}) + 6g_{\rho[\bar{\tau}_1}(BG_{\bar{\tau}_2\bar{\tau}_3]k\bar{k}} + BG_{\bar{\tau}_2\bar{\tau}_3]-+}). \quad (C.28)
\end{aligned}$$

Furthermore, \mathcal{I}_M with $M = \bar{\rho}$ is given by the expressions

$$\begin{aligned}
\Gamma^{\bar{k}} &: LG_{\bar{\rho}-} + 24BG_{k\bar{k}\bar{\rho}} - 24BG_{\bar{\rho}-\sigma}{}^\sigma, \\
\Gamma^{\bar{\tau}} &: -6i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\bar{\rho}\sigma_1\sigma_2-} - 48BG_{k\bar{\rho}\bar{\tau}-}, \\
\Gamma^+ &: 2i\epsilon^{\sigma_1\cdots\sigma_3} LF_{\bar{\rho}\sigma_1\cdots\sigma_3} + LG_{k\bar{\rho}} - 24BG_{k\bar{\rho}\sigma}{}^\sigma + 24BG_{k\bar{\rho}-+}, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -3i\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma LF_{\bar{k}\bar{\rho}\sigma-} - 12BG_{\bar{\rho}\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -3i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\bar{k}\bar{\rho}\sigma_1\sigma_2} + \frac{1}{2}LG_{\bar{\rho}\bar{\tau}} + 12(BG_{k\bar{k}\bar{\rho}\bar{\tau}} - BG_{\bar{\rho}\bar{\tau}\sigma}{}^\sigma + BG_{\bar{\rho}\bar{\tau}-+}), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\bar{\rho}-} - 12iLF_{k\bar{k}\bar{\rho}-} + 12iLF_{\bar{\rho}-\sigma}{}^\sigma), \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(E_{\bar{\rho}\sigma_1} - 12iLF_{\bar{\rho}\sigma_1 k\bar{k}} + 12iLF_{\bar{\rho}\sigma_1\sigma_2}{}^{\sigma_2} + 12iLF_{\bar{\rho}\sigma-+}) - 12BG_{k\bar{\rho}\bar{\tau}_1\bar{\tau}_2}, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{48}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\bar{k}\bar{\rho}} - 12iLF_{\bar{k}\bar{\rho}\sigma}{}^\sigma - 12iLF_{\bar{k}\bar{\rho}-+}). \quad (C.29)
\end{aligned}$$

The integrability conditions \mathcal{I}_- read

$$\begin{aligned}
\Gamma^{\bar{k}} &: 0, \\
\Gamma^{\bar{\tau}} &: 0, \\
\Gamma^+ &: -2i\epsilon^{\sigma_1\cdots\sigma_3} LF_{\sigma_1\cdots\sigma_3-} + LG_{k-} - 24BG_{k-\sigma}{}^\sigma, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: 0, \\
\Gamma^{\bar{k}\bar{\tau}+} &: -3i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\bar{k}\sigma_1\sigma_2-} - \frac{1}{2}LG_{\bar{\tau}-} - 12(BG_{k\bar{k}\bar{\tau}-} - BG_{\bar{\tau}-\sigma}{}^\sigma), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_2\cdots\bar{\tau}_3}E_{--}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(E_{\sigma_1-} + 12iLF_{k\bar{k}\sigma_1-} - 12iLF_{\sigma_1\sigma_2}{}^{\sigma_2-}) - 12BG_{k\bar{\tau}_1\bar{\tau}_2-}, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{48}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\bar{k}-} - 12iLF_{\bar{k}-\sigma}{}^\sigma) + 2BG_{\bar{\tau}_1\cdots\bar{\tau}_3-}. \quad (C.30)
\end{aligned}$$

Finally, for \mathcal{I}_+ we find

$$\begin{aligned}
\Gamma^{\bar{k}} &: -LG_{-+} + 24BG_{k\bar{k}\sigma}{}^\sigma - 12BG_{\sigma_1}{}^{\sigma_1\sigma_2}{}_{\sigma_2}, \\
\Gamma^{\bar{\tau}} &: 6i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\sigma_1\sigma_2-+} + 48BG_{k\bar{\tau}\sigma}{}^\sigma, \\
\Gamma^+ &: -2i\epsilon^{\sigma_1\cdots\sigma_3} LF_{\sigma_1\cdots\sigma_3+} + LG_{k+} + 24BG_{k+\sigma}{}^\sigma, \\
\Gamma^{\bar{k}\bar{\tau}_1\bar{\tau}_2} &: -3i\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^\sigma LF_{\bar{k}\sigma-+} + 12(BG_{k\bar{k}\bar{\tau}_1\bar{\tau}_2} - BG_{\bar{\tau}_1\bar{\tau}_2\sigma}{}^\sigma), \\
\Gamma^{\bar{k}\bar{\tau}+} &: -3i\epsilon_{\bar{\tau}}{}^{\sigma_1\sigma_2} LF_{\bar{k}\sigma_1\sigma_2+} - \frac{1}{2}LG_{\bar{\tau}+} + 12(BG_{k\bar{k}\bar{\tau}+} - BG_{\bar{\tau}+\sigma}{}^\sigma), \\
\Gamma^{\bar{\tau}_1\cdots\bar{\tau}_3} &: -\frac{1}{24}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{-+} + 12iLF_{k\bar{k}-+} - 12iLF_{\sigma}{}^\sigma{}_{-+}) + 8BG_{k\bar{\tau}_1\cdots\bar{\tau}_3}, \\
\Gamma^{\bar{\tau}_1\bar{\tau}_2+} &: \frac{1}{8}\epsilon_{\bar{\tau}_1\bar{\tau}_2}{}^{\sigma_1}(E_{\sigma_1+} + 12iLF_{k\bar{k}\sigma_1+} - 12iLF_{\sigma_1\sigma_2}{}^{\sigma_2+}) + 12BG_{k\bar{\tau}_1\bar{\tau}_2+}, \\
\Gamma^{\bar{k}\bar{\tau}_1\cdots\bar{\tau}_3+} &: \frac{1}{48}\epsilon_{\bar{\tau}_1\cdots\bar{\tau}_3}(E_{\bar{k}+} - 12iLF_{\bar{k}+\sigma}{}^\sigma) - 2BG_{\bar{\tau}_1\cdots\bar{\tau}_3+}. \quad (C.31)
\end{aligned}$$

D Generic half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds

D.1 The linear system

We decompose the vector $SO(9,1)$ representation under $SU(4)$. This is equivalent to decomposing the frame indices as $A = (+, -, \alpha, \bar{\alpha})$. Consequently, the fluxes and geometry decompose into $SU(4)$ representations, i.e. P_A decomposes as P_+, P_-, P_α and $P_{\bar{\alpha}}$ and similarly for the other fluxes and geometry¹⁵.

Next, we construct the linear system associated with the algebraic and supercovariant connection Killing spinor equations. In particular, the algebraic Killing spinor equations give

$$(A_{11} + iA_{12})P_{\bar{\alpha}} + \frac{1}{4}G_{-+\bar{\alpha}} + \frac{1}{4}G_{\bar{\alpha}\beta}{}^\beta + \frac{1}{12}\epsilon_{\bar{\alpha}}{}^{\beta_1\beta_2\beta_3}G_{\beta_1\beta_2\beta_3} = 0, \quad (\text{D.1})$$

$$(A_{21} + iA_{22})P_{\bar{\alpha}} + \frac{i}{4}G_{-+\bar{\alpha}} + \frac{i}{4}G_{\bar{\alpha}\beta}{}^\beta - \frac{i}{12}\epsilon_{\bar{\alpha}}{}^{\beta_1\beta_2\beta_3}G_{\beta_1\beta_2\beta_3} = 0, \quad (\text{D.2})$$

$$(A_{11} - iA_{12})P_\alpha + \frac{1}{4}G_{-+\alpha} - \frac{1}{4}G_{\alpha\beta}{}^\beta + \frac{1}{12}\epsilon_\alpha{}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0, \quad (\text{D.3})$$

$$(A_{21} - iA_{22})P_\alpha - \frac{i}{4}G_{-+\alpha} + \frac{i}{4}G_{\alpha\beta}{}^\beta + \frac{i}{12}\epsilon_\alpha{}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0, \quad (\text{D.4})$$

$$(A_{11} + iA_{12})P_+ + \frac{1}{4}G_{+\alpha}{}^\alpha = 0, \quad (\text{D.5})$$

$$(A_{21} + iA_{22})P_+ + \frac{i}{4}G_{+\alpha}{}^\alpha = 0, \quad (\text{D.6})$$

$$(A_{11} - iA_{12})P_+ - \frac{1}{4}G_{+\alpha}{}^\alpha = 0, \quad (\text{D.7})$$

$$(A_{21} - iA_{22})P_+ + \frac{i}{4}G_{+\alpha}{}^\alpha = 0, \quad (\text{D.8})$$

¹⁵If the fluxes are complex, like P and G , then their various components do not satisfy the ‘naive’ complex conjugate relations, i.e. $(P_\alpha)^* \neq P_{\bar{\alpha}}$ and similarly for G .

and

$$G_{+\bar{\alpha}\bar{\beta}} - \frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}}^{\gamma\delta}G_{+\gamma\delta} = 0 . \quad (\text{D.9})$$

$$iG_{+\bar{\alpha}\bar{\beta}} + \frac{i}{2}\epsilon_{\bar{\alpha}\bar{\beta}}^{\gamma\delta}G_{+\gamma\delta} = 0 , \quad (\text{D.10})$$

where we have set $A = z^{-1}z^*$.

The Killing spinor equation associated with the supercovariant derivative (2.6) also decomposes in $SU(4)$ representations. In particular the conditions associated with \mathcal{D}_α are

$$(z^{-1}D_\alpha z)_{11} + i(z^{-1}D_\alpha z)_{12} + \frac{1}{2}\Omega_{\alpha,\beta}^\beta + \frac{1}{2}\Omega_{\alpha,-+} + iF_{\alpha-+}^\beta + \frac{1}{4}(A_{11} + iA_{12})G_{\alpha\beta}^\beta + \frac{1}{4}(A_{11} + iA_{12})G_{\alpha-+} = 0 , \quad (\text{D.11})$$

$$(z^{-1}D_\alpha z)_{21} + i(z^{-1}D_\alpha z)_{22} + i[\frac{1}{2}\Omega_{\alpha,\beta}^\beta \frac{1}{2}\Omega_{\alpha,-+} + iF_{\alpha-+}^\beta] + \frac{1}{4}(A_{21} + iA_{22})[G_{\alpha\beta}^\beta + G_{\alpha-+}] = 0 , \quad (\text{D.12})$$

$$\Omega_{\alpha,\bar{\beta}_1\bar{\beta}_2} + iF_{\alpha\bar{\beta}_1\bar{\beta}_2}^\gamma + iF_{\alpha-+\bar{\beta}_1\bar{\beta}_2} + \frac{1}{2}(A_{11} + iA_{12})G_{\alpha\bar{\beta}_1\bar{\beta}_2} + (B_{11} + iB_{12})\delta_{\alpha[\bar{\beta}_1}P_{\bar{\beta}_2]} - \frac{1}{2}[\Omega_{\alpha,\gamma_1\gamma_2} + \frac{1}{2}(A_{11} - iA_{12})G_{\alpha\gamma_1\gamma_2}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{D.13})$$

$$i[\Omega_{\alpha,\bar{\beta}_1\bar{\beta}_2} + iF_{\alpha\bar{\beta}_1\bar{\beta}_2}^\gamma + iF_{\alpha-+\bar{\beta}_1\bar{\beta}_2}] + \frac{1}{2}(A_{21} + iA_{22})G_{\alpha\bar{\beta}_1\bar{\beta}_2} + (B_{21} + iB_{22})\delta_{\alpha[\bar{\beta}_1}P_{\bar{\beta}_2]} - \frac{1}{2}[-i\Omega_{\alpha,\gamma_1\gamma_2} + \frac{1}{2}(A_{21} - iA_{22})G_{\alpha\gamma_1\gamma_2}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{D.14})$$

$$(z^{-1}D_\alpha z)_{11} - i(z^{-1}D_\alpha z)_{12} + \frac{1}{2}(B_{11} - iB_{12})P_\alpha - \frac{1}{2}\Omega_{\alpha,\beta}^\beta + \frac{1}{2}\Omega_{\alpha,-+} - \frac{1}{4}(A_{11} - iA_{12})[G_{\alpha\beta}^\beta - G_{\alpha-+}] + \frac{i}{12}F_{\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}\epsilon^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} = 0 , \quad (\text{D.15})$$

$$(z^{-1}D_\alpha z)_{21} - i(z^{-1}D_\alpha z)_{22} + \frac{1}{2}(B_{21} - iB_{22})P_\alpha - i[-\frac{1}{2}\Omega_{\alpha,\beta}^\beta + \frac{1}{2}\Omega_{\alpha,-+}] - \frac{1}{4}(A_{21} - iA_{22})[G_{\alpha\beta}^\beta - G_{\alpha-+}] - \frac{1}{12}F_{\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}\epsilon^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} = 0 , \quad (\text{D.16})$$

$$\frac{1}{2}\Omega_{\alpha,+ \bar{\beta}} - \frac{1}{4}(B_{11} + iB_{12})\delta_{\alpha\bar{\beta}}P_+ + \frac{1}{4}(A_{11} + iA_{12})G_{\alpha+ \bar{\beta}} + \frac{i}{2}F_{\alpha+ \bar{\beta}}^\gamma = 0 , \quad (\text{D.17})$$

$$[\frac{i}{2}\Omega_{\alpha,+ \bar{\beta}} - \frac{1}{4}(B_{21} + iB_{22})\delta_{\alpha\bar{\beta}}P_+ + \frac{1}{4}(A_{21} + iA_{22})G_{\alpha+ \bar{\beta}} - \frac{1}{2}F_{\alpha+ \bar{\beta}}^\gamma] = 0 , \quad (\text{D.18})$$

$$\frac{i}{12}F_{\alpha+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} + \frac{1}{12}\left[\frac{1}{2}\Omega_{\alpha,+ \gamma} + \frac{1}{4}(A_{11} - iA_{12})G_{\alpha+\gamma}\right]\epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 , \quad (\text{D.19})$$

$$-\frac{1}{12}F_{\alpha+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} + \frac{1}{12}\left[-\frac{i}{2}\Omega_{\alpha,+ \gamma} + \frac{1}{4}(A_{21} - iA_{22})G_{\alpha+\gamma}\right]\epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 , \quad (\text{D.20})$$

where $B = A^2$.

The conditions associated with $\mathcal{D}_{\bar{\alpha}}$ are

$$\begin{aligned} (z^{-1}D_{\bar{\alpha}}z)_{11} + i(z^{-1}D_{\bar{\alpha}}z)_{12} + \frac{1}{2}(B_{11} + iB_{12})P_{\bar{\alpha}} + \frac{1}{2}\Omega_{\bar{\alpha},\beta}^{\beta} + \frac{1}{2}\Omega_{\bar{\alpha},-+} \\ + \frac{i}{12}F_{\bar{\alpha}\gamma_1\gamma_2\gamma_3\gamma_4}\epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} + \frac{1}{4}(A_{11} + iA_{12})[G_{\bar{\alpha}\beta}^{\beta} + G_{\bar{\alpha}-+}] = 0 , \end{aligned} \quad (\text{D.21})$$

$$\begin{aligned} (z^{-1}D_{\bar{\alpha}}z)_{21} + i(z^{-1}D_{\bar{\alpha}}z)_{22} + \frac{1}{2}(B_{21} + iB_{22})P_{\bar{\alpha}} + i\left[\frac{1}{2}\Omega_{\bar{\alpha},\beta}^{\beta} + \frac{1}{2}\Omega_{\bar{\alpha},-+} \right. \\ \left. - \frac{i}{12}F_{\bar{\alpha}\gamma_1\gamma_2\gamma_3\gamma_4}\epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4}\right] + \frac{1}{4}(A_{21} + iA_{22})[G_{\bar{\alpha}\beta}^{\beta} + G_{\bar{\alpha}-+}] = 0 , \end{aligned} \quad (\text{D.22})$$

$$\begin{aligned} \Omega_{\bar{\alpha},\bar{\beta}_1\bar{\beta}_2} + \frac{1}{2}(A_{11} + iA_{12})G_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2} \\ - \frac{1}{2}[\Omega_{\bar{\alpha},\gamma_1\gamma_2} - iF_{\bar{\alpha}\gamma_1\gamma_2\delta}^{\delta} + iF_{\bar{\alpha}-+\gamma_1\gamma_2} \\ + \frac{1}{2}(A_{11} - iA_{12})G_{\bar{\alpha}\gamma_1\gamma_2} + (B_{11} - iB_{12})\delta_{\bar{\alpha}[\gamma_1}P_{\gamma_2]}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \end{aligned} \quad (\text{D.23})$$

$$\begin{aligned} i\Omega_{\bar{\alpha},\bar{\beta}_1\bar{\beta}_2} + \frac{1}{2}(A_{21} + iA_{22})G_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2} \\ + \frac{i}{2}[\Omega_{\bar{\alpha},\gamma_1\gamma_2} - iF_{\bar{\alpha}\gamma_1\gamma_2\delta}^{\delta} + iF_{\bar{\alpha}-+\gamma_1\gamma_2}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} \\ - \frac{1}{2}\left[\frac{1}{2}(A_{21} - iA_{22})G_{\bar{\alpha}\gamma_1\gamma_2} + (B_{21} - iB_{22})\delta_{\bar{\alpha}[\gamma_1}P_{\gamma_2]}\right]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \end{aligned} \quad (\text{D.24})$$

$$\begin{aligned} (z^{-1}D_{\bar{\alpha}}z)_{11} - i(z^{-1}D_{\bar{\alpha}}z)_{12} - \frac{1}{2}\Omega_{\bar{\alpha},\gamma}^{\gamma} + \frac{1}{2}\Omega_{\bar{\alpha},-+} - iF_{\bar{\alpha}-+\gamma}^{\gamma} \\ - \frac{1}{4}(A_{11} - iA_{12})G_{\bar{\alpha}\gamma}^{\gamma} + \frac{1}{4}(A_{11} - iA_{12})G_{\bar{\alpha}-+} = 0 , \end{aligned} \quad (\text{D.25})$$

$$\begin{aligned} (z^{-1}D_{\bar{\alpha}}z)_{21} - i(z^{-1}D_{\bar{\alpha}}z)_{22} - i\left[-\frac{1}{2}\Omega_{\bar{\alpha},\gamma}^{\gamma} + \frac{1}{2}\Omega_{\bar{\alpha},-+} - iF_{\bar{\alpha}-+\gamma}^{\gamma}\right] \\ - \frac{1}{4}(A_{21} - iA_{22})G_{\bar{\alpha}\gamma}^{\gamma} + \frac{1}{4}(A_{21} - iA_{22})G_{\bar{\alpha}-+} = 0 , \end{aligned} \quad (\text{D.26})$$

$$\frac{1}{2}\Omega_{\bar{\alpha},+\bar{\beta}} + \frac{1}{4}(A_{11} + iA_{12})G_{\bar{\alpha}+\bar{\beta}} - \frac{i}{6}F_{\bar{\alpha}+\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}_{\bar{\beta}} = 0 , \quad (\text{D.27})$$

$$\frac{i}{2}\Omega_{\bar{\alpha},+\bar{\beta}} + \frac{1}{4}(A_{21} + iA_{22})G_{\bar{\alpha}+\bar{\beta}} - \frac{1}{6}F_{\bar{\alpha}+\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}_{\bar{\beta}} = 0 , \quad (\text{D.28})$$

$$\begin{aligned}
& [\frac{1}{2}\Omega_{\bar{\alpha},+\gamma} - \frac{i}{2}F_{\bar{\alpha}+\gamma\delta}^{\delta}] \epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} \\
& + \frac{1}{4}(A_{11} - iA_{12})G_{\bar{\alpha}+\gamma} \epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} - \frac{1}{4}(B_{11} - iB_{12})P_+ \epsilon_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0, \tag{D.29}
\end{aligned}$$

$$\begin{aligned}
& -i[\frac{1}{2}\Omega_{\bar{\alpha},+\gamma} - \frac{i}{2}F_{\bar{\alpha}+\gamma\delta}^{\delta}] \epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} \\
& + \frac{1}{4}(A_{21} - iA_{22})G_{\bar{\alpha}+\gamma} \epsilon^{\gamma}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} - \frac{1}{4}(B_{21} - iB_{22})P_+ \epsilon_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0. \tag{D.30}
\end{aligned}$$

The conditions associated with \mathcal{D}_- are

$$\begin{aligned}
& (z^{-1}D_-z)_{11} + i(z^{-1}D_-z)_{12} + \frac{1}{2}\Omega_{-, \gamma}^{\gamma} + \frac{1}{2}\Omega_{-, -+} + \frac{i}{4}F_{-\gamma}^{\gamma\delta} \\
& + \frac{i}{12}F_{-\gamma_1\gamma_2\gamma_3\gamma_4} \epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} + \frac{1}{4}(A_{11} + iA_{12})G_{-\gamma}^{\gamma} = 0, \tag{D.31}
\end{aligned}$$

$$\begin{aligned}
& (z^{-1}D_-z)_{21} + i(z^{-1}D_-z)_{22} + i[\frac{1}{2}\Omega_{-, \gamma}^{\gamma} + \frac{1}{2}\Omega_{-, -+} + \frac{i}{4}F_{-\gamma}^{\gamma\delta}] \\
& + \frac{1}{4}(A_{21} + iA_{22})G_{-\gamma}^{\gamma} + \frac{1}{12}F_{-\gamma_1\gamma_2\gamma_3\gamma_4} \epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} = 0, \tag{D.32}
\end{aligned}$$

$$\begin{aligned}
& \Omega_{-, \bar{\beta}_1\bar{\beta}_2} + iF_{-\bar{\beta}_1\bar{\beta}_2\gamma}^{\gamma} + \frac{1}{2}(A_{11} + iA_{12})G_{-\bar{\beta}_1\bar{\beta}_2} \\
& - \frac{1}{2}[\Omega_{-, \gamma_1\gamma_2} - iF_{-\gamma_1\gamma_2\delta}^{\delta}] \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} - \frac{1}{4}(A_{11} - iA_{12})G_{-\gamma_1\gamma_2} \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0, \tag{D.33}
\end{aligned}$$

$$\begin{aligned}
& i[\Omega_{-, \bar{\beta}_1\bar{\beta}_2} + iF_{-\bar{\beta}_1\bar{\beta}_2\gamma}^{\gamma}] + \frac{1}{2}(A_{21} + iA_{22})G_{-\bar{\beta}_1\bar{\beta}_2} \\
& + \frac{i}{2}[\Omega_{-, \gamma_1\gamma_2} - iF_{-\gamma_1\gamma_2\delta}^{\delta}] \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} - \frac{1}{4}(A_{21} - iA_{22})G_{-\gamma_1\gamma_2} \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0, \tag{D.34}
\end{aligned}$$

$$\begin{aligned}
& (z^{-1}D_-z)_{11} - i(z^{-1}D_-z)_{12} - \frac{1}{2}\Omega_{-, \gamma}^{\gamma} + \frac{1}{2}\Omega_{-, -+} + \frac{i}{4}F_{-\gamma}^{\gamma\delta} - \frac{1}{4}(A_{11} - iA_{12})G_{-\gamma}^{\gamma} \\
& + \frac{i}{12}F_{-\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} \epsilon^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} = 0, \tag{D.35}
\end{aligned}$$

$$\begin{aligned}
& (z^{-1}D_-z)_{21} - i(z^{-1}D_-z)_{22} - i[-\frac{1}{2}\Omega_{-, \gamma}^{\gamma} + \frac{1}{2}\Omega_{-, -+} + \frac{i}{4}F_{-\gamma}^{\gamma\delta}] - \frac{1}{4}(A_{21} - iA_{22})G_{-\gamma}^{\gamma} \\
& - \frac{1}{12}F_{-\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} \epsilon^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4} = 0, \tag{D.36}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\Omega_{-, +\bar{\beta}} + \frac{i}{2}F_{-+\bar{\beta}\gamma}^{\gamma} + \frac{1}{4}(A_{11} + iA_{12})G_{-+\bar{\beta}} + \frac{1}{4}(B_{11} + iB_{12})P_{\bar{\beta}} \\
& - \frac{i}{6}F_{-+\gamma_1\gamma_2\gamma_3} \epsilon^{\gamma_1\gamma_2\gamma_3}_{\bar{\beta}} = 0, \tag{D.37}
\end{aligned}$$

$$i\left[\frac{1}{2}\Omega_{-,\bar{+}\bar{\beta}} + \frac{i}{2}F_{-+\bar{\beta}\gamma}{}^{\gamma}\right] + \frac{1}{4}(A_{21} + iA_{22})G_{-+\bar{\beta}} + \frac{1}{4}(B_{21} + iB_{22})P_{\bar{\beta}} - \frac{1}{6}F_{-+\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar{\beta}} = 0 , \quad (\text{D.38})$$

$$iF_{-+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} + \left[\frac{1}{2}\Omega_{-,\bar{+}\gamma} - \frac{i}{2}F_{-+\gamma\delta}{}^{\delta}\right]\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} + \frac{1}{4}[(A_{11} - iA_{12})G_{-+\gamma} + (B_{11} - iB_{12})P_{\gamma}]\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 . \quad (\text{D.39})$$

$$-F_{-+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} - i\left[\frac{1}{2}\Omega_{-,\bar{+}\gamma} - \frac{i}{2}F_{-+\gamma\delta}{}^{\delta}\right]\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} + \frac{1}{4}[(A_{21} - iA_{22})G_{-+\gamma} + (B_{21} - iB_{22})P_{\gamma}]\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 . \quad (\text{D.40})$$

The conditions associated with \mathcal{D}_+ are

$$(z^{-1}D_+z)_{11} + i(z^{-1}D_+z)_{12} + \frac{1}{2}(B_{11} + iB_{12})P_+ + \frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+} + \frac{1}{4}(A_{11} + iA_{12})G_{+\gamma}{}^{\gamma} = 0 , \quad (\text{D.41})$$

$$(z^{-1}D_+z)_{21} + i(z^{-1}D_+z)_{22} + \frac{1}{2}(B_{21} + iB_{22})P_+ + i\left[\frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+}\right] + \frac{1}{4}(A_{21} + iA_{22})G_{+\gamma}{}^{\gamma} = 0 , \quad (\text{D.42})$$

$$\Omega_{+,\bar{\beta}_1\bar{\beta}_2} + \frac{1}{2}(A_{11} + iA_{12})G_{+\bar{\beta}_1\bar{\beta}_2} - \frac{1}{2}\Omega_{+,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} - \frac{1}{4}(A_{11} - iA_{12})G_{+\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{D.43})$$

$$i\Omega_{+,\bar{\beta}_1\bar{\beta}_2} + \frac{1}{2}(A_{21} + iA_{22})G_{+\bar{\beta}_1\bar{\beta}_2} + \frac{i}{2}\Omega_{+,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} - \frac{1}{4}(A_{21} - iA_{22})G_{+\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{D.44})$$

$$(z^{-1}D_+z)_{11} - i(z^{-1}D_+z)_{12} + \frac{1}{2}(B_{11} - iB_{12})P_+ - \frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+} - \frac{1}{4}(A_{11} - iA_{12})G_{+\gamma}{}^{\gamma} = 0 , \quad (\text{D.45})$$

$$(z^{-1}D_+z)_{21} - i(z^{-1}D_+z)_{22} + \frac{1}{2}(B_{21} - iB_{22})P_+ - i\left[-\frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+}\right] - \frac{1}{4}(A_{21} - iA_{22})G_{+\gamma}{}^{\gamma} = 0 , \quad (\text{D.46})$$

and

$$\Omega_{+,+\alpha} = \Omega_{+,\bar{\alpha}} = 0 . \quad (\text{D.47})$$

As we have already mentioned, all the equations that arise from the Killing spinor equations are linear in the fluxes, geometry and the first derivatives of the functions z that determine the Killing spinors. The system may appear involved but it can be solved. It also simplifies in some special cases, like for example whenever z is a real matrix. In this case $A = B = 1$ and so the terms in the linear system above that contain the fluxes and geometry do not depend on the functions z .

D.2 The solution to the linear system

We shall first solve the last six equations of the linear system associated with the algebraic Killing spinor equation. Equations (D.5)-(D.8) imply that

$$G_{+\alpha}{}^\alpha = P_+ = 0 , \quad (\text{D.48})$$

and (D.9), (D.10) imply that

$$G_{+\alpha\beta} = G_{+\bar{\alpha}\bar{\beta}} = 0 . \quad (\text{D.49})$$

Next consider the equations (D.19), (D.20), (D.27), (D.28). These imply

$$\Omega_{\alpha,\beta+} = 0, \quad F_{+\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 \quad (\text{D.50})$$

and (D.43), (D.44) and (D.47) imply that

$$\Omega_{+,\alpha\beta} = 0, \quad \Omega_{+,+\alpha} = 0 . \quad (\text{D.51})$$

The remaining components of the \mathcal{D}_+ equations (D.41), (D.42), (D.45), (D.46) imply that

$$\begin{aligned} (z^{-1}D_+z)_{11} &= (z^{-1}D_+z)_{22} = -\frac{1}{2}\Omega_{+,-+} \\ (z^{-1}\partial_+z)_{12} &= -(z^{-1}\partial_+z)_{21} = \frac{i}{2}\Omega_{+,\alpha}{}^\alpha . \end{aligned} \quad (\text{D.52})$$

The equations (D.31), (D.32), (D.35), (D.36) constrain $z^{-1}D_-z$ via

$$\begin{aligned} (z^{-1}D_-z)_{11} &= -\frac{1}{2}\Omega_{-,-+} - \frac{i}{4}F_{-\alpha}{}^\alpha{}_\beta{}^\beta - \frac{i}{24}(F_{-\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} \\ &\quad + F_{-\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\epsilon^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}) - \frac{i}{4}A_{12}G_{-\alpha}{}^\alpha \\ (z^{-1}D_-z)_{22} &= -\frac{1}{2}\Omega_{-,-+} - \frac{i}{4}F_{-\alpha}{}^\alpha{}_\beta{}^\beta + \frac{i}{24}(F_{-\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} \\ &\quad + F_{-\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\epsilon^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}) + \frac{i}{4}A_{21}G_{-\alpha}{}^\alpha \\ (z^{-1}\partial_-z)_{12} &= \frac{i}{2}\Omega_{-,\alpha}{}^\alpha - \frac{1}{24}((F_{-\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} - F_{-\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\epsilon^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}) \end{aligned}$$

$$\begin{aligned}
& + \frac{i}{4} A_{11} G_{-\alpha}{}^\alpha \\
(z^{-1} \partial_- z)_{21} &= -\frac{i}{2} \Omega_{-, \alpha}{}^\alpha - \frac{1}{24} ((F_{-\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} - F_{-\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4} \epsilon^{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4}) \\
& - \frac{i}{4} A_{22} G_{-\alpha}{}^\alpha .
\end{aligned} \tag{D.53}$$

And from (D.33) and (D.34), we find

$$\begin{aligned}
G_{-\bar{\alpha}_1 \bar{\alpha}_2} &= -\frac{(A_{11} + A_{22} + i(A_{21} - A_{12}))}{A_{11} A_{22} - A_{12} A_{21}} (\Omega_{-, \bar{\alpha}_1 \bar{\alpha}_2} + i F_{-\bar{\alpha}_1 \bar{\alpha}_2 \gamma}{}^\gamma) \\
&+ \frac{(A_{22} - A_{11} + i(A_{12} + A_{21}))}{2(A_{11} A_{22} - A_{12} A_{21})} (\Omega_{-, \beta_1 \beta_2} - i F_{-\beta_1 \beta_2 \gamma}{}^\gamma) \epsilon_{\bar{\alpha}_1 \bar{\alpha}_2}{}^{\beta_1 \beta_2} \\
G_{-\alpha_1 \alpha_2} &= -\frac{(A_{11} + A_{22} - i(A_{21} - A_{12}))}{A_{11} A_{22} - A_{12} A_{21}} (\Omega_{-, \alpha_1 \alpha_2} - i F_{-\alpha_1 \alpha_2 \gamma}{}^\gamma) \\
&- \frac{(A_{11} - A_{22} + i(A_{12} + A_{21}))}{2(A_{11} A_{22} - A_{12} A_{21})} (\Omega_{-, \bar{\beta}_1 \bar{\beta}_2} + i F_{-\bar{\beta}_1 \bar{\beta}_2 \gamma}{}^\gamma) \epsilon_{\alpha_1 \alpha_2}{}^{\bar{\beta}_1 \bar{\beta}_2} .
\end{aligned} \tag{D.54}$$

In order to proceed we shall consider two separate cases, according as $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2$ vanishes or not. Observe that $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$ is equivalent to $A_{11} = A_{22}$ and $A_{12} + A_{21} = 0$.

First we shall assume that $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0$. Then (D.17), (D.18), (D.29), (D.30) imply that

$$G_{+\alpha \bar{\beta}} = \Omega_{\alpha, \bar{\beta}+} = 0 \tag{D.55}$$

and

$$F_{+\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2} = 0 . \tag{D.56}$$

Next consider the equations which have one free holomorphic index, excluding for the moment those equations involving $z^{-1} D z$; these are (D.3), (D.4), the trace of the duals of (D.13), (D.14) and the duals of (D.39), (D.40). These fix P_α , $G_{-+\alpha}$, $G_{\alpha \beta}{}^\beta$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} G_{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3}$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} F_{-+\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3}$, $F_{-+\alpha \beta}{}^\beta$ in terms of the components of the spin connection $\Omega_{\bar{\beta}}{}^\beta{}_\alpha$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} \Omega_{\bar{\beta}_1, \bar{\beta}_2 \bar{\beta}_3}$, $\Omega_{-, +\alpha}$ and A_{ij} .

The corresponding equations with one free antiholomorphic constraint are (D.1), (D.4), the traces of (D.13), (D.14) and (D.37), (D.38). These fix $P_{\bar{\alpha}}$, $G_{-+\bar{\alpha}}$, $G_{\bar{\alpha} \beta}{}^\beta$, $\epsilon_{\bar{\alpha}}{}^{\beta_1 \beta_2 \beta_3} G_{\beta_1 \beta_2 \beta_3}$, $\epsilon_{\bar{\alpha}}{}^{\beta_1 \beta_2 \beta_3} F_{-+\beta_1 \beta_2 \beta_3}$, $F_{-+\bar{\alpha} \beta}{}^\beta$ in terms of the components of the spin connection $\Omega_{\beta}{}^\beta{}_{\bar{\alpha}}$, $\epsilon_{\bar{\alpha}}{}^{\beta_1 \beta_2 \beta_3} \Omega_{\beta_1, \beta_2 \beta_3}$, $\Omega_{-, +\bar{\alpha}}$.

By comparing the complex expressions for $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} F_{-+\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3}$, $F_{-+\alpha \beta}{}^\beta$ from the former equations with the complex conjugates of the expressions $\epsilon_{\bar{\alpha}}{}^{\beta_1 \beta_2 \beta_3} F_{-+\beta_1 \beta_2 \beta_3}$, $F_{-+\bar{\alpha} \beta}{}^\beta$ from the latter, we find the following geometric constraints

$$\begin{aligned}
\Omega_{\beta, \bar{\alpha}}{}^\beta &= -\frac{(6A_{11}A_{22} - 4A_{12}A_{21} - A_{12}^2 - A_{21}^2)}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-, +\bar{\alpha}} \\
\epsilon_{\bar{\alpha}}{}^{\beta_1 \beta_2 \beta_3} \Omega_{\beta_1, \beta_2 \beta_3} &= -2i \frac{(A_{12} - A_{21})(A_{11} - A_{22} + i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-, +\bar{\alpha}} .
\end{aligned} \tag{D.57}$$

Using these constraints, we obtain the following simplifications

$$\begin{aligned}
P_{\bar{\alpha}} &= \frac{4}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \\
G_{-+\bar{\alpha}} &= -8 \frac{(A_{11} + A_{22})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \\
G_{\bar{\alpha}\beta}^{\beta} &= -8i \frac{(A_{12} - A_{21})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \\
\epsilon_{\bar{\alpha}}^{\beta_1\beta_2\beta_3} G_{\beta_1\beta_2\beta_3} &= -24 \frac{(A_{11} - A_{22} + i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \quad (D.58)
\end{aligned}$$

and

$$\begin{aligned}
P_{\alpha} &= \frac{4}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\alpha} \\
G_{-+\alpha} &= -8 \frac{(A_{11} + A_{22})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\alpha} \\
G_{\alpha\beta}^{\beta} &= -8i \frac{(A_{12} - A_{21})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\alpha} \\
\epsilon_{\alpha}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} &= -24 \frac{(A_{11} - A_{22} - i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\alpha} \quad (D.59)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_{\bar{\alpha}}^{\beta_1\beta_2\beta_3} F_{-+\beta_1\beta_2\beta_3} &= -3i \frac{(A_{11} + A_{22})(A_{11} - A_{22} + i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \\
F_{-+\bar{\alpha}\beta}^{\beta} &= \frac{(A_{11} + A_{22})(A_{12} - A_{21})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-,+\bar{\alpha}} \quad (D.60)
\end{aligned}$$

Substituting these constraints back into (D.21), (D.22), (D.25) and (D.26) we find the following constraints on $z^{-1}D_{\bar{\alpha}}z$:

$$\begin{aligned}
(z^{-1}D_{\bar{\alpha}}z)_{11} &= -\frac{1}{2}\Omega_{\bar{\alpha},-+} \\
&\quad - \frac{1}{2} \frac{(A_{22}^2 - 3A_{11}^2 + 4A_{12}^2 - 4A_{11}A_{22} - 2A_{12}A_{21})}{((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))} \Omega_{-,+\bar{\alpha}} \\
(z^{-1}D_{\bar{\alpha}}z)_{22} &= -\frac{1}{2}\Omega_{\bar{\alpha},-+} \\
&\quad + \frac{1}{2} \frac{(3A_{22}^2 - A_{11}^2 - 4A_{21}^2 + 4A_{11}A_{22} + 2A_{12}A_{21})}{((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))} \Omega_{-,+\bar{\alpha}} \\
(z^{-1}\partial_{\bar{\alpha}}z)_{12} &= \frac{i}{2}\Omega_{\bar{\alpha},\beta}^{\beta} \\
&\quad + \frac{i}{2}((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))^{-1}(A_{11}^2 + A_{22}^2 - 2iA_{12}A_{22} \\
&\quad - 2iA_{21}A_{22} - 6iA_{11}A_{12} + 2iA_{11}A_{21} + 2A_{12}A_{21})\Omega_{-,+\bar{\alpha}} \\
(z^{-1}\partial_{\bar{\alpha}}z)_{21} &= -\frac{i}{2}\Omega_{\bar{\alpha},\beta}^{\beta}
\end{aligned}$$

$$\begin{aligned}
& - \frac{i}{2}((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))^{-1}(A_{11}^2 + A_{22}^2 + 6iA_{21}A_{22} \\
& - 2iA_{12}A_{22} + 2iA_{11}A_{12} + 2iA_{11}A_{21} + 2A_{12}A_{21})\Omega_{-,+\bar{\alpha}} .
\end{aligned} \tag{D.61}$$

And from (D.11), (D.12), (D.15) and (D.16) we find the following constraints on $z^{-1}D_\alpha z$:

$$\begin{aligned}
(z^{-1}D_\alpha z)_{11} &= -\frac{1}{2}\Omega_{\alpha,-+} \\
&+ \frac{1}{2} \frac{(3A_{11}^2 - A_{22}^2 - 4A_{12}^2 + 4A_{11}A_{22} + 2A_{12}A_{21})}{((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))} \Omega_{-,+\alpha} \\
(z^{-1}D_\alpha z)_{22} &= -\frac{1}{2}\Omega_{\alpha,-+} \\
&- \frac{1}{2} \frac{(A_{11}^2 - 3A_{22}^2 + 4A_{21}^2 - 4A_{11}A_{22} - 2A_{12}A_{21})}{((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))} \Omega_{-,+\alpha} \\
(z^{-1}\partial_\alpha z)_{12} &= \frac{i}{2}\Omega_{\alpha,\beta}^\beta \\
&- \frac{i}{2}((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))^{-1}(A_{11}^2 + A_{22}^2 - 2iA_{11}A_{21} \\
&+ 6iA_{11}A_{12} + 2iA_{21}A_{22} + 2iA_{12}A_{22} + 2A_{12}A_{21})\Omega_{-,+\alpha} \\
(z^{-1}\partial_\alpha z)_{21} &= -\frac{i}{2}\Omega_{\alpha,\beta}^\beta \\
&+ \frac{i}{2}((A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21}))^{-1}(A_{11}^2 + A_{22}^2 - 2iA_{11}A_{21} \\
&- 2iA_{11}A_{12} + 2iA_{12}A_{22} - 6iA_{21}A_{22} + 2A_{12}A_{21})\Omega_{-,+\alpha} .
\end{aligned} \tag{D.62}$$

Note that by means of an appropriate $U(1)$ transformation, one can work in a gauge for which $\det z = \det z^*$, so that $\det A = 1$. Then (D.61), (D.62) and (D.52) imply that

$$Q_+ = Q_\alpha = 0 \tag{D.63}$$

though in general (D.53) does not constrain Q_- to vanish.

Finally, we consider the equations (D.13), (D.14), (D.23), (D.24). The constraints obtained from taking traces of these equations have already been obtained; the remaining constraints consist of the fixing of two components of the G -flux via

$$\begin{aligned}
G_{\alpha\bar{\beta}_1\bar{\beta}_2} &= \frac{2}{A_{11} - A_{22} + i(A_{12} + A_{21})} \Omega_{\alpha,\gamma_1\gamma_2} \epsilon_{\bar{\beta}_1\bar{\beta}_2}^{\gamma_1\gamma_2} \\
&- \frac{8i(A_{21} - A_{12})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{-,+[\bar{\beta}_1\delta_{\bar{\beta}_2]\alpha} \\
G_{\bar{\alpha}\gamma_1\gamma_2} &= \frac{2}{A_{11} - A_{22} - i(A_{12} + A_{21})} \Omega_{\bar{\alpha},\bar{\beta}_1\bar{\beta}_2} \epsilon^{\bar{\beta}_1\bar{\beta}_2}_{\gamma_1\gamma_2} \\
&- \frac{8i(A_{12} - A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{-,+[\gamma_1\delta_{\gamma_2]}\bar{\alpha}
\end{aligned} \tag{D.64}$$

together with the fixing of a component of the F -flux

$$F_{-+\alpha\bar{\beta}_1\bar{\beta}_2} = (A_{11} + A_{22}) \left(\frac{i}{4(A_{11} - A_{22} + i(A_{12} + A_{21}))} \Omega_{\alpha,\gamma_1\gamma_2} \epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} \right)$$

$$- \frac{(A_{12} - A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{-,+[\bar{\beta}_1 \delta_{\bar{\beta}_2] \alpha)} \quad (\text{D.65})$$

and a geometric constraint

$$\begin{aligned} & \Omega_{\alpha, \bar{\beta}_1 \bar{\beta}_2} + \frac{i(A_{12} - A_{21})}{2(A_{11} - A_{22} + i(A_{12} + A_{21}))} \Omega_{\alpha, \gamma_1 \gamma_2} \epsilon^{\gamma_1 \gamma_2}{}_{\bar{\beta}_1 \bar{\beta}_2} \\ & - \frac{4(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{-,+[\bar{\beta}_1 \delta_{\bar{\beta}_2] \alpha} = 0. \end{aligned} \quad (\text{D.66})$$

Next we consider the special case when $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$. Then (D.17), (D.18), (D.29), (D.30) imply that

$$G_{+\alpha \bar{\beta}} = \frac{1}{A_{11}} (\Omega_{\alpha, +\bar{\beta}} - \Omega_{\bar{\beta}, +\alpha}) \quad (\text{D.67})$$

$$F_{+\alpha \bar{\beta} \gamma}{}^{\gamma} = -\frac{i}{2A_{11}} ((A_{11} - iA_{12})\Omega_{\alpha, +\bar{\beta}} + (A_{11} + iA_{12})\Omega_{\bar{\beta}, +\alpha}). \quad (\text{D.68})$$

On comparing (D.68) with its complex conjugate, the geometric constraint

$$\Omega_{\alpha, +\bar{\beta}} + \Omega_{\bar{\beta}, +\alpha} = 0 \quad (\text{D.69})$$

is obtained. Substituting this back into (D.67) and (D.68) we obtain some simplification:

$$\begin{aligned} G_{+\alpha \bar{\beta}} &= \frac{2}{A_{11}} \Omega_{\alpha, +\bar{\beta}} \\ F_{+\alpha \bar{\beta} \gamma}{}^{\gamma} &= -\frac{A_{12}}{A_{11}} \Omega_{\alpha, +\bar{\beta}}. \end{aligned} \quad (\text{D.70})$$

Next consider the equations which have one free holomorphic index, excluding for the moment those equations involving $z^{-1}Dz$; these are (D.3), (D.4), the trace of the duals of (D.13), (D.14) and the duals of (D.39), (D.40). Unlike the generic case, these do not fix P_α , $G_{-\alpha}$, $G_{\alpha\beta}{}^\beta$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} G_{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3}$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} F_{-\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3}$, $F_{-\alpha\beta}{}^\beta$ uniquely in terms of the components of the spin connection $\Omega_{\bar{\beta}, \bar{\beta}}{}^\beta{}_\alpha$, $\epsilon_\alpha{}^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} \Omega_{\bar{\beta}_1, \bar{\beta}_2 \bar{\beta}_3}$, $\Omega_{-, +\alpha}$ and A_{ij} . Rather, $G_{\bar{\alpha}\beta}{}^\beta$, $F_{-\bar{\alpha}\beta}{}^\beta$ and $\Omega_{\beta, \bar{\beta}}{}^\beta{}_{\bar{\alpha}}$ are arbitrary with

$$\begin{aligned} P_{\bar{\alpha}} &= -\frac{1}{3(A_{11} + iA_{12})} G_{\bar{\alpha}\beta}{}^\beta - \frac{2}{3(A_{11} + iA_{12})^2} (\Omega_{\beta, \bar{\beta}}{}^\beta{}_{\bar{\alpha}} - iF_{-\bar{\alpha}\beta}{}^\beta) \\ G_{-\bar{\alpha}} &= \frac{1}{3} G_{\bar{\alpha}\beta}{}^\beta + \frac{8}{3(A_{11} + iA_{12})} (\Omega_{\beta, \bar{\beta}}{}^\beta{}_{\bar{\alpha}} - iF_{-\bar{\alpha}\beta}{}^\beta) \\ G_{\alpha_1 \alpha_2 \alpha_3} &= 0 \\ F_{-\alpha_1 \alpha_2 \alpha_3} &= 0 \\ \Omega_{[\alpha_1, \alpha_2 \alpha_3]} &= 0 \\ \Omega_{-, +\bar{\alpha}} &= -\Omega_{\beta, \bar{\beta}}{}^\beta{}_{\bar{\alpha}}. \end{aligned} \quad (\text{D.71})$$

The corresponding equations with one free antiholomorphic constraint are (D.1), (D.4), the traces of (D.13), (D.14) and (D.37), (D.38). Taking $G_{\alpha\beta}{}^\beta$, $F_{-\alpha\beta}{}^\beta$ and $\Omega_{\bar{\beta},\bar{\alpha}}{}^\beta$ to be arbitrary we find the additional constraints

$$\begin{aligned} P_\alpha &= \frac{1}{3(A_{11} - iA_{12})} G_{\alpha\beta}{}^\beta - \frac{2}{3(A_{11} - iA_{12})^2} (\Omega_{\bar{\beta},\bar{\alpha}}{}^\beta + iF_{-\alpha\beta}{}^\beta) \\ G_{-\alpha} &= -\frac{1}{3} G_{\alpha\beta}{}^\beta + \frac{8}{3(A_{11} - iA_{12})} (\Omega_{\bar{\beta},\bar{\alpha}}{}^\beta + iF_{-\alpha\beta}{}^\beta) \\ G_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3} &= 0. \end{aligned} \quad (\text{D.72})$$

Substituting these constraints back into (D.21), (D.22), (D.25) and (D.26) we find the following constraints on $z^{-1}D_{\bar{\alpha}}z$:

$$\begin{aligned} (z^{-1}D_{\bar{\alpha}}z)_{11} &= (z^{-1}D_{\bar{\alpha}}z)_{22} = -\frac{i}{6}A_{12}G_{\bar{\alpha}\beta}{}^\beta - \frac{1}{2}\Omega_{\bar{\alpha},-+} \\ &+ \frac{1}{6(A_{11} + iA_{12})}(-3A_{11} + iA_{12})\Omega_{\beta,\bar{\alpha}}{}^\beta \\ &+ \frac{i}{3(A_{11} + iA_{12})}(3A_{11} + iA_{12})F_{-+\bar{\alpha}\beta}{}^\beta \\ (z^{-1}\partial_{\bar{\alpha}}z)_{12} &= -(z^{-1}\partial_{\bar{\alpha}}z)_{21} = \frac{i}{2}\Omega_{\bar{\alpha},\beta}{}^\beta + \frac{i}{6}A_{11}G_{\bar{\alpha}\beta}{}^\beta \\ &+ \frac{i}{6(A_{11} + iA_{12})}(-A_{11} + 3iA_{12})\Omega_{\beta,\bar{\alpha}}{}^\beta \\ &- \frac{2A_{11}}{3(A_{11} + iA_{12})}F_{-+\bar{\alpha}\beta}{}^\beta. \end{aligned} \quad (\text{D.73})$$

And from (D.11), (D.12), (D.15) and (D.16) we find the following constraints on $z^{-1}D_\alpha z$:

$$\begin{aligned} (z^{-1}D_\alpha z)_{11} &= (z^{-1}D_\alpha z)_{22} = -\frac{1}{2}\Omega_{\alpha,-+} - \frac{i}{6}A_{12}G_{\alpha\beta}{}^\beta \\ &- \frac{1}{6(A_{11} - iA_{12})}(3A_{11} + iA_{12})\Omega_{\bar{\beta},\alpha}{}^\beta \\ &+ \frac{i}{3(A_{11} - iA_{12})}(-3A_{11} + iA_{12})F_{-\alpha\beta}{}^\beta \\ (z^{-1}\partial_\alpha z)_{12} &= -(z^{-1}\partial_\alpha z)_{21} = \frac{i}{2}\Omega_{\alpha,\beta}{}^\beta + \frac{i}{6}A_{11}G_{\alpha\beta}{}^\beta \\ &+ \frac{i}{6(A_{11} - iA_{12})}(A_{11} + 3iA_{12})\Omega_{\bar{\beta},\alpha}{}^\beta \\ &- \frac{2A_{11}}{3(A_{11} - iA_{12})}F_{-\alpha\beta}{}^\beta. \end{aligned} \quad (\text{D.74})$$

Finally, we consider the equations (D.13), (D.14), (D.23), (D.24). These fix two components of the G -flux via

$$G_{\alpha\bar{\beta}_1\bar{\beta}_2} = -\frac{2}{A_{11} + iA_{12}}(\Omega_{\alpha,\bar{\beta}_1\bar{\beta}_2} + 2iF_{-\alpha\bar{\beta}_1\bar{\beta}_2})$$

$$\begin{aligned}
& - \frac{1}{A_{11} + iA_{12}} \delta_{\alpha[\bar{\beta}_1} \left(-\frac{2}{3}(A_{11} + iA_{12})G_{\bar{\beta}_2]\gamma}{}^\gamma - \frac{4}{3}\Omega_{[\gamma],\gamma}{}^{\bar{\beta}_2]} - \frac{8i}{3}F_{\bar{\beta}_2] - + \gamma}{}^\gamma \right) \\
G_{\bar{\alpha}\gamma_1\gamma_2} & = -\frac{2}{A_{11} - iA_{12}} (\Omega_{\bar{\alpha},\gamma_1\gamma_2} + 2iF_{-+\bar{\alpha}\gamma_1\gamma_2}) \\
& - \frac{1}{A_{11} - iA_{12}} \delta_{\bar{\alpha}[\gamma_1} \left(\frac{2}{3}(A_{11} - iA_{12})G_{\gamma_2]\beta}{}^\beta - \frac{4}{3}\Omega_{[\beta],\bar{\beta}}{}^{\gamma_2]} + \frac{8i}{3}F_{\gamma_2] - + \beta}{}^\beta \right) \quad (D.75)
\end{aligned}$$

together with the geometric constraint

$$\Omega_{\alpha_1, \alpha_2 \alpha_3} = 0. \quad (D.76)$$

As the first order equations in spacetime derivatives of z are non-linear, we shall not investigate the general case here. Instead we shall focus on two examples that illustrate some of the properties of this non-linear system.

D.3 Special cases

We shall first consider the case that z is diagonal with complex entries. We have seen that it can be arranged such that the Killing spinors can be written as $\epsilon_1 = \rho e^{i\varphi} \eta_1$ and $\epsilon_1 = \rho^{-1} e^{-i\varphi} \eta_2$. Using this, the equations (D.61) and (D.62) imply that

$$\begin{aligned}
\Omega_{\alpha, \beta}{}^\beta &= \frac{\cos 4\varphi}{2 + \cos 4\varphi} \Omega_{-, +\alpha} \\
Q_\alpha &= 0 \\
\Omega_{\alpha, -+} + \Omega_{-, \alpha+} &= 0 \quad (D.77)
\end{aligned}$$

and

$$\begin{aligned}
\partial_\alpha \rho &= 0 \\
\partial_\alpha \varphi + \frac{\sin 4\varphi}{2 + \cos 4\varphi} \Omega_{-, +\alpha} &= 0. \quad (D.78)
\end{aligned}$$

From (D.53) we obtain

$$\begin{aligned}
Q_- &= \frac{1}{2} F_{-\alpha}{}^\alpha{}_\beta{}^\beta \\
\Omega_{-, \alpha}{}^\alpha &= -\frac{1}{2} \cos 2\varphi G_{-\alpha}{}^\alpha \\
F_{-\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} - F_{-\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4} \epsilon^{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4} &= 6 \sin 2\varphi G_{-\alpha}{}^\alpha \\
\Omega_{-, -+} &= 0 \quad (D.79)
\end{aligned}$$

and

$$\begin{aligned}
\partial_- \varphi &= -\frac{1}{24} (F_{-\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} - F_{-\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4} \epsilon^{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4}) \\
\partial_- \rho &= 0. \quad (D.80)
\end{aligned}$$

From (D.52) we obtain

$$\begin{aligned}\Omega_{+,\alpha}{}^\alpha &= 0 \\ Q_+ &= 0 \\ \Omega_{+,-+} &= 0\end{aligned}\tag{D.81}$$

and

$$\begin{aligned}\partial_+\varphi &= 0 \\ \partial_+\rho &= 0.\end{aligned}\tag{D.82}$$

For the other example we take z to be real and so A is the identity matrix. Then from (D.73) we find

$$\begin{aligned}Q_\alpha &= 2F_{-+\alpha\beta}{}^\beta \\ F_{-+\alpha\beta}{}^\beta &= \frac{i}{8}(G_{\alpha\beta}{}^\beta + (G_{\bar{\alpha}\beta}{}^\beta)^*)\end{aligned}\tag{D.83}$$

and

$$\begin{aligned}(z^{-1}\partial_\alpha z)_{11} &= (z^{-1}\partial_\alpha z)_{22} = -\frac{1}{2}\Omega_{\alpha,-+} - \frac{1}{2}\Omega_{\bar{\beta},\alpha}{}^{\bar{\beta}} \\ (z^{-1}\partial_\alpha z)_{12} &= -(z^{-1}\partial_\alpha z)_{21} = \frac{i}{2}\Omega_{\alpha,\beta}{}^\beta + \frac{i}{6}\Omega_{\bar{\beta},\alpha}{}^{\bar{\beta}} + \frac{i}{12}(G_{\alpha\beta}{}^\beta - (G_{\bar{\alpha}\beta}{}^\beta)^*) .\end{aligned}\tag{D.84}$$

From (D.53) we obtain

$$\begin{aligned}Q_- &= \frac{1}{2}F_{-\alpha}{}^\alpha{}_\beta{}^\beta \\ F_{-\alpha_1\alpha_2\alpha_3\alpha_4} &= 0 \\ (G_{-\alpha}{}^\alpha)^* + G_{-\alpha}{}^\alpha &= 0\end{aligned}\tag{D.85}$$

and

$$\begin{aligned}(z^{-1}\partial_- z)_{11} &= (z^{-1}\partial_- z)_{22} = -\frac{1}{2}\Omega_{-,-+} \\ (z^{-1}\partial_- z)_{12} &= -(z^{-1}\partial_- z)_{21} = \frac{i}{2}\Omega_{-,\alpha}{}^\alpha + \frac{i}{4}G_{-\alpha}{}^\alpha .\end{aligned}\tag{D.86}$$

From (D.52) we obtain

$$Q_+ = 0\tag{D.87}$$

and

$$\begin{aligned}(z^{-1}\partial_+ z)_{11} &= (z^{-1}\partial_+ z)_{22} = -\frac{1}{2}\Omega_{+,-+} \\ (z^{-1}\partial_+ z)_{12} &= -(z^{-1}\partial_+ z)_{21} = \frac{i}{2}\Omega_{+,\alpha}{}^\alpha .\end{aligned}\tag{D.88}$$

E The linear system of degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$ -backgrounds

It is straightforward to construct the linear system for the degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$ case using the linear system of [12] for one $SU(4) \ltimes \mathbb{R}^8$ -invariant spinor and those of section 6. In particular, we have that the algebraic Killing spinor equations give

$$\begin{aligned} (f + g_2 - ig_1)P_{\bar{\alpha}} &= 0, \\ \frac{1}{4}(f - g_2 + ig_1)G_{-+\bar{\alpha}} + \frac{1}{4}(f - g_2 + ig_1)G_{\bar{\alpha}\beta}{}^{\beta} \\ + \frac{1}{12}(f + g_2 + ig_1)\epsilon_{\bar{\alpha}}{}^{\beta_1\beta_2\beta_3}G_{\beta_1\beta_2\beta_3} &= 0, \end{aligned} \quad (\text{E.1})$$

$$\begin{aligned} (f - g_2 - ig_1)P_{\alpha} &= 0, \\ \frac{1}{4}(f + g_2 + ig_1)G_{-+\alpha} - \frac{1}{4}(f + g_2 + ig_1)G_{\alpha\beta}{}^{\beta} \\ + \frac{1}{12}(f - g_2 + ig_1)\epsilon_{\alpha}{}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} &= 0, \end{aligned} \quad (\text{E.2})$$

$$\begin{aligned} (f + g_2 - ig_1)P_{+} &= 0, \\ \frac{1}{4}(f - g_2 + ig_1)G_{+\alpha}{}^{\alpha} &= 0, \end{aligned} \quad (\text{E.3})$$

$$\begin{aligned} (f - g_2 - ig_1)P_{+} &= 0, \\ -\frac{1}{4}(f + g_2 + ig_1)G_{+\alpha}{}^{\alpha} &= 0, \end{aligned} \quad (\text{E.4})$$

and

$$(f - g_2 + ig_1)G_{+\bar{\alpha}\bar{\beta}} - \frac{1}{2}(f + g_2 + ig_1)\epsilon_{\bar{\alpha}\bar{\beta}}{}^{\gamma\delta}G_{+\gamma\delta} = 0. \quad (\text{E.5})$$

The conditions arising from the \mathcal{D}_{α} component of the supercovariant derivative are

$$\begin{aligned} [D_{\alpha} + (w - w^*)^{-1}\partial_{\alpha}w + \frac{1}{2}\Omega_{\alpha,\beta}{}^{\beta} + \frac{1}{2}\Omega_{\alpha,-+} + \frac{i}{4}F_{\alpha\beta}{}^{\beta}{}^{\gamma} \\ + \frac{i}{2}F_{\alpha-+}{}^{\beta}{}^{\beta}] (f - g_2 + ig_1) = 0, \end{aligned} \quad (\text{E.6})$$

$$-(w - w^*)^{-1}\partial_{\alpha}w(f - g_2 + ig_1) + (f + g_2 - ig_1)\left[\frac{1}{4}G_{\alpha\beta}{}^{\beta} + \frac{1}{4}G_{-+\alpha}\right] = 0, \quad (\text{E.7})$$

$$\begin{aligned} (f - g_2 + ig_1)[\Omega_{\alpha,\bar{\beta}_1\bar{\beta}_2} + iF_{\alpha\bar{\beta}_1\bar{\beta}_2}{}^{\gamma} + iF_{\alpha-+\bar{\beta}_1\bar{\beta}_2}] \\ -(f + g_2 + ig_1)\left[\frac{1}{2}\Omega_{\alpha,\gamma_1\gamma_2} - \frac{i}{2}F_{\alpha\gamma_1\gamma_2}{}^{\delta} + \frac{i}{2}F_{\alpha-+\gamma_1\gamma_2}\right]\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0, \end{aligned} \quad (\text{E.8})$$

$$(f + g_2 - ig_1)\left[\frac{1}{2}G_{\alpha\bar{\beta}_1\bar{\beta}_2} - \frac{1}{4}g_{\alpha[\bar{\beta}_1}G_{\bar{\beta}_2]\gamma}{}^{\gamma} - \frac{1}{4}g_{\alpha[\bar{\beta}_1}G_{\bar{\beta}_2]-+}\right]$$

$$-\frac{1}{8}(f - g_2 - ig_1)G_{\alpha\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{E.9})$$

$$\begin{aligned} & [D_\alpha + (w - w^*)^{-1}\partial_\alpha w - \frac{1}{2}\Omega_{\alpha,\beta}{}^\beta + \frac{1}{2}\Omega_{\alpha,-+} + \frac{i}{4}F_{\alpha\beta}{}^\beta{}_\gamma{}^\gamma \\ & - \frac{i}{2}F_{\alpha-+}{}^\gamma](f + g_2 + ig_1) + \frac{i}{12}F_{\alpha\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}\epsilon^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}(f - g_2 + ig_1) = 0 , \end{aligned} \quad (\text{E.10})$$

$$\begin{aligned} & -(w - w^*)^{-1}\partial_\alpha w(f + g_2 + ig_1) + [-\frac{1}{8}G_{\alpha\gamma}{}^\gamma + \frac{1}{8}G_{-+\alpha}](f - g_2 - ig_1) \\ & - \frac{1}{24}\epsilon_\alpha{}^{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}(f + g_2 - ig_1) = 0 , \end{aligned} \quad (\text{E.11})$$

$$[\frac{1}{2}\Omega_{\alpha,+}{}^{\bar{\beta}} + \frac{i}{2}F_{\alpha+}{}^{\bar{\beta}}{}_\gamma{}^\gamma](f - g_2 + ig_1) - \frac{i}{6}F_{\alpha+\beta_1\beta_2\beta_3}\epsilon^{\beta_1\beta_2\beta_3}_{\bar{\beta}}(f + g_2 + ig_1) = 0 , \quad (\text{E.12})$$

$$[\frac{1}{16}g_{\alpha\bar{\beta}}G_{+\gamma}{}^\gamma - \frac{1}{4}G_{+\alpha\bar{\beta}}](f + g_2 - ig_1) = 0 , \quad (\text{E.13})$$

$$\frac{i}{12}F_{\alpha+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}(f - g_2 + ig_1) + \frac{1}{12}(f + g_2 + ig_1)[\frac{1}{2}\Omega_{\alpha,+}{}^\gamma - \frac{i}{2}F_{\alpha+\gamma}{}^\delta] \epsilon^\gamma_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 , \quad (\text{E.14})$$

$$\frac{1}{32}g_{\alpha[\bar{\beta}_1}G_{\bar{\beta}_2\bar{\beta}_3]+}(f + g_2 - ig_1) - \frac{1}{96}(f - g_2 - ig_1)G_{+\alpha\gamma}\epsilon^\gamma_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 . \quad (\text{E.15})$$

The conditions arising from the $\mathcal{D}_{\bar{\alpha}}$ component of the supercovariant derivative are

$$\begin{aligned} & [D_{\bar{\alpha}} + (w - w^*)^{-1}\partial_{\bar{\alpha}} w + \frac{1}{2}\Omega_{\bar{\alpha},\beta}{}^\beta + \frac{1}{2}\Omega_{\bar{\alpha},-+} + \frac{i}{4}F_{\bar{\alpha}\beta}{}^\beta{}_\gamma{}^\gamma \\ & + \frac{i}{2}F_{\bar{\alpha}-+}{}^\beta](f - g_2 + ig_1) + \frac{i}{12}(f + g_2 + ig_1)F_{\bar{\alpha}\gamma_1\gamma_2\gamma_3\gamma_4}\epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} = 0 , \end{aligned} \quad (\text{E.16})$$

$$\begin{aligned} & -(w - w^*)^{-1}\partial_{\bar{\alpha}} w(f - g_2 + ig_1) + \frac{1}{8}[G_{\bar{\alpha}\beta}{}^\beta + G_{\bar{\alpha}-+}](f + g_2 - ig_1) \\ & - \frac{1}{24}(f - g_2 - ig_1)\epsilon_{\bar{\alpha}}{}^{\gamma_1\gamma_2\gamma_3}G_{\gamma_1\gamma_2\gamma_3} = 0 , \end{aligned} \quad (\text{E.17})$$

$$\begin{aligned} & [\Omega_{\bar{\alpha},\bar{\beta}_1\bar{\beta}_2} + iF_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2}{}^\gamma + iF_{\bar{\alpha}-+}{}^{\bar{\beta}_1\bar{\beta}_2}](f - g_2 + ig_1) \\ & - (f + g_2 + ig_1)[\frac{1}{2}\Omega_{\bar{\alpha},\gamma_1\gamma_2} - \frac{i}{2}F_{\bar{\alpha}\gamma_1\gamma_2}{}^\delta + \frac{i}{2}F_{\bar{\alpha}-+}{}^{\gamma_1\gamma_2}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \end{aligned} \quad (\text{E.18})$$

$$\begin{aligned} & \frac{1}{4}G_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2}(f + g_2 - ig_1) - (f - g_2 - ig_1)[\frac{1}{8}g_{\bar{\alpha}\gamma_1}G_{\gamma_2}{}^\delta \\ & + \frac{1}{4}G_{\bar{\alpha}\gamma_1\gamma_2} - \frac{1}{8}g_{\bar{\alpha}\gamma_1}G_{\gamma_2-+}]\epsilon^{\gamma_1\gamma_2}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \end{aligned} \quad (\text{E.19})$$

$$\begin{aligned}
& [D_{\bar{\alpha}} + (w - w^*)^{-1} \partial_{\bar{\alpha}} w - \frac{1}{2} \Omega_{\bar{\alpha}, \gamma}{}^{\gamma} + \frac{1}{2} \Omega_{\bar{\alpha}, -+} \\
& + \frac{i}{4} F_{\bar{\alpha} \gamma}{}^{\gamma \delta} - \frac{i}{2} F_{\bar{\alpha} - + \gamma}{}^{\gamma}] (f + g_2 + i g_1) = 0 , \tag{E.20}
\end{aligned}$$

$$-(w - w^*)^{-1} \partial_{\bar{\alpha}} w (f + g_2 + i g_1) + [-\frac{1}{4} G_{\bar{\alpha} \gamma}{}^{\gamma} + \frac{1}{4} G_{\bar{\alpha} - +}] (f - g_2 - i g_1) = 0 , \tag{E.21}$$

$$\begin{aligned}
& [\frac{1}{2} \Omega_{\bar{\alpha}, + \bar{\beta}} + \frac{i}{2} F_{\bar{\alpha} + \bar{\beta} \gamma}{}^{\gamma}] (f - g_2 + i g_1) \\
& - \frac{i}{6} (f + g_2 + i g_1) F_{\bar{\alpha} + \gamma_1 \gamma_2 \gamma_3} \epsilon^{\gamma_1 \gamma_2 \gamma_3}{}_{\bar{\beta}} = 0 , \tag{E.22}
\end{aligned}$$

$$-\frac{1}{8} G_{+ \bar{\alpha} \bar{\beta}} (f + g_2 - i g_1) - \frac{1}{16} (f - g_2 - i g_1) G_{\gamma_1 \gamma_2 +} \epsilon^{\gamma_1 \gamma_2}{}_{\bar{\alpha} \bar{\beta}} = 0 , \tag{E.23}$$

$$i F_{\bar{\alpha} + \bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} (f - g_2 + i g_1) + (f + g_2 + i g_1) [\frac{1}{2} \Omega_{\bar{\alpha}, + \gamma} - \frac{i}{2} F_{\bar{\alpha} + \gamma \delta}{}^{\delta}] \epsilon^{\gamma}{}_{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} = 0 , \tag{E.24}$$

$$(f - g_2 - i g_1) [-\frac{1}{16} g_{\bar{\alpha} \gamma} G_{+ \delta}{}^{\delta} + \frac{1}{4} G_{\bar{\alpha} + \gamma}] \epsilon^{\gamma}{}_{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3} = 0 . \tag{E.25}$$

The conditions arising from the \mathcal{D}_- component of the supercovariant derivative are

$$\begin{aligned}
& [D_- + (w - w^*)^{-1} \partial_- w + \frac{1}{2} \Omega_{-, \gamma}{}^{\gamma} + \frac{1}{2} \Omega_{-, -+} + \frac{i}{4} F_{- \gamma}{}^{\gamma \delta}] (f - g_2 + i g_1) \\
& + \frac{i}{12} (f + g_2 + i g_1) F_{- \gamma_1 \gamma_2 \gamma_3 \gamma_4} \epsilon^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} = 0 , \tag{E.26}
\end{aligned}$$

$$-(w - w^*)^{-1} \partial_- w (f - g_2 + i g_1) + \frac{1}{4} G_{- \gamma}{}^{\gamma} (f + g_2 - i g_1) = 0 , \tag{E.27}$$

$$\begin{aligned}
& [\Omega_{-, \bar{\beta}_1 \bar{\beta}_2} + i F_{- \bar{\beta}_1 \bar{\beta}_2 \gamma}{}^{\gamma}] (f - g_2 + i g_1) \\
& - (f + g_2 + i g_1) [\frac{1}{2} \Omega_{-, \gamma_1 \gamma_2} - \frac{i}{2} F_{- \gamma_1 \gamma_2 \delta}{}^{\delta}] \epsilon^{\gamma_1 \gamma_2}{}_{\bar{\beta}_1 \bar{\beta}_2} = 0 , \tag{E.28}
\end{aligned}$$

$$\frac{1}{2} G_{- \bar{\beta}_1 \bar{\beta}_2} (f + g_2 - i g_1) - \frac{1}{4} (f - g_2 - i g_1) G_{- \gamma_1 \gamma_2} \epsilon^{\gamma_1 \gamma_2}{}_{\bar{\beta}_1 \bar{\beta}_2} = 0 , \tag{E.29}$$

$$\begin{aligned}
& [D_- + (w - w^*)^{-1} \partial_- w - \frac{1}{2} \Omega_{-, \gamma}{}^{\gamma} + \frac{1}{2} \Omega_{-, -+} + \frac{i}{4} F_{- \gamma}{}^{\gamma \delta}] (f + g_2 + i g_1) \\
& + \frac{i}{12} (f - g_2 + i g_1) F_{- \bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3 \bar{\beta}_4} \epsilon^{\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3 \bar{\beta}_4} = 0 , \tag{E.30}
\end{aligned}$$

$$-(w - w^*)^{-1} \partial_- w (f + g_2 + i g_1) - \frac{1}{4} (f - g_2 - i g_1) G_{- \gamma}{}^{\gamma} = 0 , \tag{E.31}$$

$$[\frac{1}{2}\Omega_{-,+\bar{\beta}} + \frac{i}{2}F_{-+\bar{\beta}\gamma}{}^{\gamma}](f - g_2 + ig_1) - \frac{i}{6}(f + g_2 + ig_1)F_{-+\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar{\beta}} = 0 , \quad (\text{E.32})$$

$$[-\frac{1}{16}G_{\bar{\beta}\gamma}{}^{\gamma} + \frac{3}{16}G_{-+\bar{\beta}}](f + g_2 - ig_1) + \frac{1}{48}(f - g_2 - ig_1)G_{\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar{\beta}} = 0 , \quad (\text{E.33})$$

$$iF_{-+\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}(f - g_2 + ig_1) + [\frac{1}{2}\Omega_{-,+\gamma} - \frac{i}{2}F_{-+\gamma\delta}{}^{\delta}]\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}(f + g_2 + ig_1) = 0 . \quad (\text{E.34})$$

$$-\frac{1}{8}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}(f + g_2 - ig_1) + [\frac{1}{16}G_{\gamma\delta}{}^{\delta} + \frac{3}{16}G_{-+\gamma}](f - g_2 - ig_1)\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 . \quad (\text{E.35})$$

The conditions arising from the \mathcal{D}_+ component of the supercovariant derivative are

$$[D_+ + (w - w^*)^{-1}\partial_+ w + \frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+} + \frac{i}{4}F_{+\gamma}{}^{\gamma\delta}](f - g_2 + ig_1) = 0 , \quad (\text{E.36})$$

$$-(w - w^*)^{-1}\partial_+ w(f - g_2 + ig_1) + \frac{1}{8}G_{+\gamma}{}^{\gamma}(f + g_2 - ig_1) = 0 , \quad (\text{E.37})$$

$$[\Omega_{+,\bar{\beta}_1\bar{\beta}_2} + iF_{+\bar{\beta}_1\bar{\beta}_2\gamma}{}^{\gamma}](f - g_2 + ig_1) - [\frac{1}{2}\Omega_{+,\gamma_1\gamma_2} - \frac{i}{2}F_{+\gamma_1\gamma_2\delta}{}^{\delta}](f + g_2 + ig_1)\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{E.38})$$

$$\frac{1}{4}(f + g_2 - ig_1)G_{+\bar{\beta}_1\bar{\beta}_2} - \frac{1}{8}(f - g_2 - ig_1)G_{+\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0 , \quad (\text{E.39})$$

$$[D_+ + (w - w^*)^{-1}\partial_+ w - \frac{1}{2}\Omega_{+,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{+,-+} + \frac{i}{4}F_{+\gamma}{}^{\gamma\delta}](f + g_2 + ig_1) = 0 , \quad (\text{E.40})$$

$$-(w - w^*)^{-1}\partial_+ w(f + g_2 + ig_1) - \frac{1}{8}G_{+\gamma}{}^{\gamma}(f - g_2 - ig_1) = 0 , \quad (\text{E.41})$$

and

$$\Omega_{+,+\alpha} = \Omega_{+,+\bar{\alpha}} = 0 . \quad (\text{E.42})$$

The solution of the above linear system and the conditions on the geometry and the fluxes are summarized in section 6.

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